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A Detailed Recipe

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Quantifying the Efficiency of the Xetra LOB Market A Detailed Recipe*

Miriam Sperl[1]

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Abstract:
Motivated by the prominent role of electronic limit order book (LOB) markets in today's stock market environment, this paper provides the basis for understanding, reconstructing and adopting Hollifield, Miller, Sandas, and Slive's (2006) (henceforth HMSS) methodology for estimating the gains from trade to the Xetra LOB market at the Frankfurt Stock Exchange (FSE) in order to evaluate its performance in this respect. Therefore this paper looks deeply into HMSS's base model and provides a structured recipe for the planned implementation with Xetra LOB data. The contribution of this paper lies in the modification of HMSS's methodology with respect to the particularities of the Xetra trading system that are not yet considered in HMSS's base model. The necessary modifications, as expressed in terms of empirical caveats, are substantial to derive unbiased market efficiency measures for Xetra in the end.

JEL Classification: G10, G14

Keywords: Limit Order Book Market, Gains from Trade, Xetra

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1 Introduction

The enormous progress in computer and telecommunication technologies accompanied by the dramatic decline in their development costs as observed over the last few decades have promoted and still keep on promoting the diffusion of electronic open limit order book (LOB) markets across the world. By the end of the nineties Domowitz and Steil (1999) not only observe that many emerging markets apply the LOB design right from the very beginning but their analysis also reveals that a significant number of already existing markets undergo restructuring away from traditional trading floors to pure order driven or hybrid trading systems.[1] To underline this development, they report that in the exemplary time span from 1997 to 1998 a considerable number of sixteen stock exchanges, including as prominent examples London, Tokyo and Toronto, conduct such a transformation. A recent study provided by Jain (2003) shows that nowadays the overwhelming majority of over 80% of the world's exchanges with a market capitalization of equal dimension operate some sort of electronic trading mechanism with automatic execution.[2]

Unlike traditional exchange mechanisms open LOB markets operate without the intermediation of dealers, who participate in every trade in quote driven markets, or brokers, who are responsible for finding liquidity in brokered markets. Instead LOB markets use rule-based order matching systems that are implemented electronically and in which liquidity supply and demand are provided by traders themselves instead of dealers or brokers: patient traders offer liquidity by indicating the conditions under which they will trade via the submission of instructions to buy (sell) a given number of shares at the best price possible but not to pay (demand) more (less) than a limit price (limit order traders). Impatient traders directly demand liquidity by accepting the limit order traders' conditions via the submission of market orders that trade a given number of shares at the best price available (market order traders). The limit orders are queued

[1] Hybrid trading systems obey characteristics of quote driven, brokered and order driven markets. The currently most prominent example of a hybrid market is the New York Stock Exchange.

[2] More precisely 51% of the world's exchanges operate as pure LOB markets with a share of market capitalization of 28% while the other 29% of the world's exchanges are attributable to hybrid markets with a share of market capitalization of 50%.

into the electronic LOB according to order precedence rules while well defined pricing rules ascertain their automatic execution. In open LOB markets the electronic order book is displayed to all market participants.

Given the prominent role of the LOB market design in today's stock market environment, an important question is how efficient LOB markets actually operate and whether particular LOB markets perform better than others. For such a performance evaluation it is essential to obtain a profound knowledge of the determinants of liquidity and price formation in LOB markets. The analysis of the latter in turn requires a deep understanding of the fundamental decision problem that traders face in LOB markets involving not only the order type choice (choice between buy versus sell order and market versus limit order) but also the timing of the order submission.

A vast literature developed around these subjects. Two classes of theoretical LOB models derive equilibrium prices in LOB markets and exploit how price setting rules evolve and determine traders' optimal order placement strategies: the class of static and the class of dynamic LOB models. The former are static in the sense that equilibrium conditions in LOB markets are determined at a specific point in time considering only past and current information. In contrast, the equilibrium conditions in dynamic LOB models are determined over time, i.e. not only by taking into account current and past information, but also by allowing for the anticipation of future events. The roots of these model classes go back to three seminal papers: the work of Glosten (1994) who introduced the class of static LOB models and the models of Parlour (1998) and Foucault (1999) who particularly influenced the class of dynamic LOB models. The theoretical LOB models reveal as driving forces of the price formation and the liquidity provision in LOB markets the optimal order placement strategies of traders that result from finding trade-offs between execution probabilities, picking off risks and order prices for alternative order submissions. Order book information turns out to considerably help traders to solve this fundamental decision problem.

On the basis of the predictions provided by the theoretical LOB models a variety of descriptive and econometric analyses evolved that deal with econometric modeling and hypothesis testing: Sandås (2001) and Frey and Grammig (2006) present econometric methodologies for testing economic restrictions on the price schedules offered in pure open LOB markets. Their approaches directly build on Glosten's (1994) baseline model and incorporate real world LOB market features like discrete prices and time priority rules. While Sandås's (2001) approach contains too many simplifying assumptions rendering impossible to fit the data well, Frey and Grammig's (2006) variant succeeds to provide evidence for Glosten's (1994) model putting forward one of the key messages of LOB market theory, namely that liquidity supply and adverse selection costs are inversely related.

In common with Biais, Hillion, and Spatt (1995) who analyze the interaction between the order book and the order flow using a sample from the LOB market at the Paris Bourse by means of rather basic statistical tools, a variety of descriptive and experimental studies use the rich information flow delivered by electronic LOB markets to test predictions of dynamic LOB market theory and to reveal further stylized facts (Degryse, de Jong, Ravenswaaij, and Wuyts (2005), Coppejans, Domowitz, and Madhavan (2004), Gomber, Schweickert, and Theissen (2004) and Cao, Hansch, and Wang (2006)). Besides the assessment of the informational content of the order book in determining a security's true value, the main focus of these studies lies on the analysis of aggressive orders and the resiliency of LOB markets. The aggressiveness of an order is measured in terms of the price and time priority demanded by traders; the higher the demand for price and time priority of a trader the more aggressive his order. The resiliency of a market refers to the rate at which prices revert to former levels after having been changed in response to large order flow imbalances induced by aggressive orders (cp. Harris (2003)) and is used to evaluate an LOB market's performance in this respect.

While the studies in line with Biais, Hillion, and Spatt (1995) were mainly descriptive, Griffiths, Smith, Turnbull, and White (2000), Ranaldo (2004) and Pascual and Veredas (2006) implement the classification of orders according to their aggressiveness by means of ordered probit techniques with explanatory variables that capture the state of the order book. Bisière and Kamionka (2000), Hall and Hautsch (2006) and Large (2007) argue in favor of using multivariate point processes (duration or intensity based models) that allow to model market orders, limit orders and cancelations as individual but interdependent processes while simultaneously accounting for order book dynamics. The previous literature of econometric LOB models provides evidence for the intertwined dynamics between the order book and the order flow in real world LOB markets. It further points out the significant informational content of the LOB with regard to the underlying value of assets. Moreover, the literature presents means for the performance evaluation of real world LOB markets on the basis of the various dimensions of liquidity putting forward results that confirm the quality of this market design.

Instead of measuring a market's quality in terms of its resiliency, which is only one dimension of liquidity, the calculation of the gains agents derive from trading provides another way to assess a market's quality. Economists measure the gains agents derive from trading by the surpluses they obtain as a consequence of a transaction. A seller's surplus is generally defined as the difference between the trade price and his valuation of the stock, a buyer's surplus as the difference between his valuation of the stock and the trade price, with both surpluses eventually being reduced by transaction costs. What makes it challenging to measure the gains from trade by trader surpluses is that, in general, the valuation a trader places on the stock he trades is unknown.

Hollifield, Miller, Sandås, and Slive (2006) (henceforth HMSS) provide an econometric methodology that links traders' optimal order submissions in LOB markets with traders' valuations for the stock and the trade-offs across execution probabilities, picking off risks and order prices for alternative order submissions compare Hollifield, Miller, and Sandås (2004). These relations make possible to determine traders' surpluses and to actually compute estimates of the gains from trade. The main end-product of HMSS's approach are such estimates derived for an order driven stock exchange that are computed as a percentage of the theoretical benchmarks of maximum possible or monopoly-induced gains from trade - results that serve as standardized and hence comparable measures to assess the efficiency of real world LOB markets. HMSS's methodology provides convincing results for the estimates of the gains from trade for the LOB market at the Vancouver Stock Exchange (VSE) - a fact that encourages to adopt their method to other real world LOB markets for the purpose of market performance evaluation and comparison.[3] The objective of this paper is to look deeply into HMSS's methodology for estimating the gains from trade in LOB markets in order to provide a detailed recipe for the planned application of this method to the Xetra LOB system at the Frankfurt Stock Exchange (FSE).

The remainder of this paper is organized as follows. Section 2 explicitly describes the market structure of the Xetra trading system at the FSE which provides useful for the adoption of HMSS's methodology to the German stock market as it reveals potential features of the Xetra LOB market that need to be factored into the HMSS base model. The econometric methodology for quantifying the efficiency of the Xetra LOB market in terms of the gains from trade is presented in section 3. Section 4 provides a detailed recipe for the econometric implementation and the final computation of HMSS's market efficiency measures. Section 5 deals with empirical caveats and extensions. Section 6 finally concludes.

2 Market Structure of the Xetra LOB Market

Trading takes place during the trading session that can be specified further by its trading forum, its trading hours and by the type of trading session that is used to arrange trades. Concerning the trading forum, the Xetra trading system is a so-called distributed access market: due to client-server solutions traders in different locations have worldwide access to this system via their trading screens. With respect to the trading hours, regular trading on Xetra takes place during the normal business hours from 9:00 a.m. CET to 5:30 p.m. CET on trading days of the FSE. During the pretrading phase starting at 7:30 a.m. CET market participants are allowed to cancel and change old or submit new orders and quotes to prepare for the main trading phase. The hours after the closing of the main trading phase until 8:30 p.m. CET are used to adjust existing positions and to

[3]HMSS find that for the LOB market at the VSE the current gains from trade are approximately 90% of the maximum gains from trade and approximately 50% more than the monopoly gains from trade.

submit new orders that will be incorporated in the next day's regular trading session. Moreover, traders use these posttrading hours to work on their trades with regard to the settlement and the reporting of the results to potential clients. Regarding the type of trading session that is used to arrange trades, Xetra can behave as both a continuous market and a call market depending on the security that needs to be traded. While only a negligible fraction of securities[4] is traded in a call auction once a day or in continuous auctions throughout the day, for the majority of securities, such as the stocks of the DAX 30 for example, Xetra is generally referred to as a continuous market in which trading is possible anytime the market is open. Nevertheless, the regular continuous trading session on Xetra is enriched by call market elements since it begins with an opening call auction, ends with a closing call auction and is interrupted by a mid-day call auction. Besides, call auctions can be used to restart continuous trading after trading halts.

The execution system of a market constitutes the core of the market structure as it defines how buyers are matched to sellers and how trade prices are determined. The three main types of execution systems are the quote driven market, the brokered market and the order driven market. The electronic trading system Xetra is a typical example of the latter containing traces of a quote driven market for smaller listed stocks.[5] In order to do without the intermediation of dealers, who participate in every trade in the quote driven market, or brokers, who are responsible for finding liquidity in the brokered market, Xetra uses a rule-based order-matching system. This system ascertains a smooth functioning of the trading procedure via a set of well defined trading rules that are implemented electronically and work without further intermediation under normal circumstances: the order precedence rules that match buyers to sellers and the pricing rules that define which price has to be paid for a particular trade. In order to take part in the trading process, traders make their trading conditions available to the Xetra system in electronic form. As a result, liquidity supply and demand are provided

[4]Namely 'other shares' and 'Covered Warrants, Certificates, Reverse Convertibles'; see the Xetra® trading parameters available on the Deutsche Börse webpage (http://deutsche-boerse.com/dbag/dispatch/de/kir/gdb_navigation/trading_members/12_Xetra/45_Trading_Parameter).

[5]If a stock does not comply with the liquidity requirements for continuous trading, it needs to be supervised by a so-called designated sponsor. The banks or securities trading houses, who act as designated sponsors, guarantee price quality by providing additional liquidity.

by traders themselves as limit order traders offer liquidity by indicating the conditions under which they will trade and market order traders directly demand liquidity by accepting the limit order traders' conditions. Nevertheless, trading on Xetra is not directly allowed to any individual who wants to trade but is rather delimited to financial institutions and securities trading houses that are simply referred to as traders from now on. Individuals can only trade indirectly through these channels.

Xetra's primary order precedence rule is price priority meaning that the trader, who is willing to buy at the highest price among all buyers and the trader, who is willing to sell at the lowest price among all sellers, are matched first. If the primary order precedence rule is insufficient to unambiguously rank all buyers and sellers, the secondary order precedence rule finds a remedy which is time priority for Xetra: for two orders being identically ranked along the price dimension, an order will be considered first if it has been submitted first. In the presence of special order types like iceberg orders time priority can be dominated by the display precedence rule that gives the visible fraction of the order precedence over its hidden counterpart.[6] The interplay of Xetra's order precedence rules encourages traders to permanently improve the best prices and to be honest concerning the display of their orders which are features that are very beneficial for traders who wish to execute immediately.

Once an unambiguous hierarchy of buy and sell orders is assessed with the help of order precedence rules and once all this information is stored in the electronic order book, sellers are matched with buyers along this hierarchy. Trades occur whenever the sellside overlaps the buyside in the sense that sellers meet buyers that accept the respective trade conditions concerning the price and the volume of the asset of interest. In continuous markets the best bid and the best offer do not overlap but trades rather occur immediately whenever incoming orders can be filled with already existing orders stored in the order book. Orders that can be filled immediately are called marketable orders, in contrast to non-marketable orders that enter the order book and still have to wait for being executed.

[6] In the case of iceberg orders (also called hidden orders) only a fraction of the order quantity is displayed to the market.

After matching buyers with sellers, trade prices need to be determined. The determination of trade prices on Xetra follows two different rules. As Xetra is both a continuous market as well as a call market depending on the security that needs to be traded, this feature is reflected in its execution system, too: when behaving as a continuous market, Xetra conducts continuous two-sided auctions and the price discovery process follows the so-called discriminatory pricing rule; when behaving as a call market, single-price auctions are required for the price discovery process alongside the so-called uniform pricing rule. Under the discriminatory pricing rule, it is the limit price of the standing order that determines the price for each trade, whereas under the uniform pricing rule all trades take place at the same market clearing price being the price that maximizes traded volume.[7]

Concerning the market information system, the existence of an electronic order book on Xetra ensures that all information related to orders is stored with the maximum possible accuracy: a market order is stored with its date, time and volume at entry and its date, time, volume and price at execution; a limit order is stored with its date, time, volume and limit price at entry and its time at (partial) execution or cancelation; and so on. Hence, Xetra's electronic order book is more than simply an information collection system, but it is an extremely valuable source of information as it reveals the conditions under which trades occur. During continuous trading Xetra behaves as an open LOB market in the sense that its order book is displayed to all market participants. Open LOB markets offer the highest degree of transparency as they do not only report quotes and orders immediately, which is referred to as offering ex ante transparency, but also report trades without any delay, which is referred to as offering ex post transparency. Xetra is less transparent during single price auctions where the order book is partially closed. The amount of information announced during single price auctions varies dynamically depending on the market situation: while in the case of a crossed order book the hypothetical price resulting under the uniform pricing rule is published, in the case

[7] For a more detailed description and exemplary illustration see Marktmodell Aktien, Xetra® Release 7.1, available at the webpage of Deutsche Börse (http://deutsche-boerse.com/dbag/dispatch/de/kir/gdb_navigation/trading_members/12_Xetra/35_Market_Model).

of non-overlapping buy and sell sides the best bid and ask quotes are announced, both indications eventually enriched by information about market imbalances.[8] Nevertheless, traders do not favor the highest degree of transparency in all respects: although they wish to see all information available about the behavior of other traders, they usually prefer to act in secrecy in order to maintain potential informational advantages. Xetra handles this fact as follows: it provides ex ante anonymity for all instruments, but ex post anonymity only where Xetra has a central counterparty service (CCP). For various instruments such as those of the DAX for example, the CCP arbitrates between sellers and buyers and does not only ensure anonymity, but also takes the potential risk of default.

3 Methodology for Quantifying the Efficiency of the Xetra LOB Market

3.1 Theoretical Model

The basic setup of the model can be summarized as follows: the execution system underlying HMSS's analysis is a continuous order driven market for a single risky asset. The asset's underlying value y_t is a random variable with innovations drawn from a stationary process with possibly time-varying conditional moments. The market dispenses with designated market makers and works on the basis of an electronic open LOB with a potential multiple-tick spread between the best quotes at the best ask price $p_{t,0}^{buy}$ and the best bid price $p_{t,0}^{sell}$. The trading process is regulated by strict price and time priority as order precedence rules. The determination of trade prices follows the discriminatory pricing rule. The market is further characterized as a one-shot, one-unit market.[9] Orders may last for multiple periods and can be canceled but are not permitted to be modified. The submission of an order requires the payment of an order submission fee $c_0 \geq 0$,

[8]For further details see Marktmodell Aktien, Xetra® Release 7.1.

[9]The one-unit characteristic is simply to reduce notation. In principle the model can deal with multiple-unit orders, too.

while the execution of an order involves an order execution cost $c_e \geq 0$.

Regarding the characteristics of agents, HMSS make the following assumptions: agents are risk-neutral and endowed only with public information contained in the information set $z_t = (x_t, \omega_t)$, where z_t follows a stationary Markov process and x_t and ω_t denote finite-dimensional vectors of exogenous and endogenous state variables.[10] The endogenous state variables ω_t are chosen as to predict the outcomes of order submissions in t, while the purpose of the exogenous state variables x_t is to predict the distribution of the common value innovations introduced above as well as the trader arrival rates and the private value distribution of traders presented in the following. Traders arrive sequentially with a conditional trader arrival rate equal to

$$\lim_{\Delta t \to 0} \frac{\Pr(\text{Trader arrives in } [t, t + \Delta t) \mid x_t)}{\Delta t} = \lambda(t; x_t). \tag{1}$$

They differ in their valuations v_t of the stock, which are decomposed as follows

$$v_t = y_t + u_t, \tag{2}$$

where y_t is the above introduced underlying value of the stock and u_t denotes a trader specific private valuation of the stock drawn from the conditional distribution

$$\Pr(u_t \leq u \mid x_t) \equiv G(u \mid x_t) \tag{3}$$

with its corresponding density denoted as $g(u|x_t)$. Once trader t arrived at the market, his valuation v_t remains fixed and he chooses between buy or sell market order submissions and a variety of buy or sell limit order submissions. His order choice is formalized with the help of decision indicators: $d_{t,s}^{sell} \in \{0,1\}$ with the finite set of available sell order submissions $s = 0, 1, \ldots, S$ and $d_{t,b}^{buy} \in \{0,1\}$ with the finite set of available buy order submissions $b = 0, 1, \ldots, B$. If $d_{t,s}^{sell} = 0$ for all s this signals that no sell order is submitted at t. The time t submission of a sell market order at price $p_{t,0}^{sell}$ is indicated by $d_{t,0}^{sell} = 1$,

[10] A Markov process obeys the so-called Markov property that implies that the future of the process, given the present, is independent of the past (cp. Spanos (1986)).

while $d^{sell}_{t,s} = 1$ represents the time t submission of a sell limit order at price $p^{sell}_{t,s}$, a price s ticks above the current best bid quote. The buy side indicator works similarly.

Trades exhibit the following characteristics: a limit order submitted at time t either executes at the random execution time $t + \tau_{execute}$ or cancels at the random cancelation time $t + \tau_{cancel}$. The cancelation of an order does not occur later than $t + \Delta T$, i.e. τ_{cancel} is bounded from above by ΔT. The execution of an order requires that $\tau_{execute} \leq \tau_{cancel}$ and the order is canceled otherwise which can be summarized with the help of the indicator function

$$I_t(\tau_{execute} \leq \tau_{cancel}) = \begin{cases} 1, & \text{if } t + \tau_{execute} \leq t + \tau_{cancel}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The conditional distributions of the latent execution and cancelation times illustrated for the case of an arbitrary buy limit order

$$\Pr(t + \tau_{execute} \leq t + \tau \mid z_t, d^{buy}_{t,b} = 1) = F_{execute}(\tau \mid z_t, d^{buy}_{t,b} = 1), \tag{5}$$

$$\Pr(t + \tau_{cancel} \leq t + \tau \mid z_t, d^{buy}_{t,b} = 1) = F_{cancel}(\tau \mid z_t, d^{buy}_{t,b} = 1) \tag{6}$$

show that the limit order's outcome (execution or cancelation) is uncertain and depends on the state vector z_t as well as on the trader's order submission. As a consequence of this uncertainty, the trader submitting the above presented arbitrary buy limit order faces the risk of non-execution represented by the execution probability

$$\psi^{buy}_b(z_t) \equiv \mathbb{E}[I_t(\tau_{execute} \leq \tau_{cancel}) \mid z_t, d^{buy}_{t,b} = 1], \tag{7}$$

and the picking off risk

$$\xi^{buy}_b(z_t) \equiv \mathbb{E}[I_t(\tau_{execute} \leq \tau_{cancel})(y_{t+\tau_{execute}} - y_t) \mid z_t, d^{buy}_{t,b} = 1], \tag{8}$$

both conditional on the state vector z_t and the order submission. The latter can be expressed in terms of the conditional execution probability as follows

$$\xi_b^{buy}(z_t) = \mathbb{E}[(y_{t+\tau_{execute}} - y_t) \mid I_t(\tau_{execute} \leq \tau_{cancel}) = 1, z_t, d_{t,b}^{buy} = 1]\psi_b^{buy}(z_t).^{11} \quad (9)$$

The trader's order submission choice depends on the utility he realizes as a consequence of the submission of a particular order. In the case of the exemplary buy order the trader's realized utility is given by

$$\begin{aligned} & I_t(\tau_{execute} \leq \tau_{cancel})(y_{t+\tau_{execute}} + u_t - p_{t,b}^{buy} - c_e) - c_0 \\ = \; & I_t(\tau_{execute} \leq \tau_{cancel})(y_t + u_t - p_{t,b}^{buy} - c_e) \\ & + I_t(\tau_{execute} \leq \tau_{cancel})(y_{t+\tau_{execute}} - y_t) - c_0, \end{aligned} \quad (10)$$

where the term on the second line reflects the payoff that the trader would earn at immediate execution and the term on the third line is the payoff due to a change in the common value minus the order submission cost. In the case of non-execution the realized utility is reduced to $-c_0$. Using the definitions of the execution probability and the picking off risk, the expected utility from submitting a buy order at price $p_{t,b}^{buy}$ as a function of the trader's valuation $y_t + u_t$ is given by

$$U_b^{buy}(y_t + u_t; z_t) = \psi_b^{buy}(z_t)(y_t + u_t - p_{t,b}^{buy} - c_e) + \xi_b^{buy}(z_t) - c_0. \quad (11)$$

The expected utility from submitting a sell order at price $p_{t,s}^{sell}$ is derived similarly and looks as follows

$$U_s^{sell}(y_t + u_t; z_t) = \psi_s^{sell}(z_t)(p_{t,s}^{sell} - y_t - u_t - c_e) - \xi_s^{sell}(z_t) - c_0. \quad (12)$$

Given a realization of his valuation of the stock $y_t + u_t$ and given a certain state of the world z_t, trader t behaves as expected utility maximizer in order to determine his

[11] This expression is a result of an application of the law of iterated expectation: $\mathbb{E}[\mathbb{E}[\,.\,|\mathcal{F}_1]|\mathcal{F}_0] = \mathbb{E}[\,.\,|\mathcal{F}_0]$, where the information sets $\mathcal{F}_1 = [z_t, d_{t,b}^{buy}]$ and $\mathcal{F}_0 = [I_t(\tau_{execute} \leq \tau_{cancel}) = 1, z_t, d_{t,b}^{buy}]$. The additional condition on order execution is obvious since the picking off risk only hurts if the order is executed.

optimal order submission strategy. He solves the following representative optimization problem:

$$\max_{\{d_{t,s}^{sell}\},\{d_{t,b}^{buy}\}} \sum_{s=0}^{S} d_{t,s}^{sell} U_s^{sell}(y_t + u_t; z_t) + \sum_{b=0}^{B} d_{t,b}^{buy} U_b^{buy}(y_t + u_t; z_t), \tag{13}$$

subject to

$$\sum_{s=0}^{S} d_{t,s}^{sell} + \sum_{b=0}^{B} d_{t,b}^{buy} \leq 1, \tag{14}$$

where equation (14) reflects the one-shot market characteristic. Denote $\{d_s^{sell*}(y_t + u_t; z_t), d_b^{buy*}(y_t + u_t; z_t)\}$ as this optimization problem's solutions. Considering the whole population of expected utility maximizing traders who have different valuations for the stock, the randomness of $y_t + u_t$ spans a whole set of potential optimal order submissions: $\mathcal{S}^*(z_t) = \{s_0(z_t), s_1(z_t), \ldots, s_S(z_t)\}$ and $\mathcal{B}^*(z_t) = \{b_0(z_t), b_1(z_t), \ldots, b_B(z_t)\}$ with indices sorted by decreasing execution probability. This set of potential optimal order submissions is a subset of the set of available order submissions reduced by those order submissions that are not optimal for anybody.[12]

Hollifield, Miller, and Sandås (2004) show that the traders' optimal order submission strategies can be represented by threshold valuations, i.e. valuations of the stock that mark the limits of intervals within which specific order types are submitted. In other words, traders with valuations ranging in the same interval submit orders of the same type. Hollifield, Miller, and Sandås (2004) further show that optimal order submissions exhibit a monotonicity property in the sense that traders with extremely low private valuations submit sell orders with high execution probabilities while traders with extremely high private valuations submit buy orders with high execution probabilities. The less extreme the trader's private valuation the less probable the execution of the order submitted such that traders with intermediate private valuations either submit no order or limit orders with low execution probabilities. The monotonicity property in its complete elegance ensures that the threshold valuations form a monotone sequence

[12] Remember that the set of available order submissions was described by the decision indicators $d_{t,s}^{sell} \in \{0,1\}$ for $s = 0, 1, \ldots, S$ and $d_{t,b}^{buy} \in \{0,1\}$ for $b = 0, 1, \ldots, B$, where s and b index the finite set of available order submissions.

and that order submissions are uniquely related to threshold intervals, a property that is useful for the subsequent construction of the market efficiency measures in terms of the gains from trade.[13]

The computation of such threshold valuations is based on pairwise expected utility comparisons for alternative order submissions. The thresholds, denoted by θ, are the valuations associated with indifference between two possible order submissions. Consider e.g. a sell order submitted at price $p^{sell}_{t,s_{i-1}(z_t)}$ and a sell order submitted at price $p^{sell}_{t,s_i(z_t)}$, then a trader is indifferent between both orders if

$$\begin{aligned}
U^{sell}_{s_{i-1}(z_t)}(y_t + u_t; z_t) &= U^{sell}_{s_i(z_t)}(y_t + u_t; z_t) \\
\psi^{sell}_{s_{i-1}(z_t)}(z_t)(p^{sell}_{t,s_{i-1}(z_t)} - y_t - u_t - c_e) &= \psi^{sell}_{s_i(z_t)}(z_t)(p^{sell}_{t,s_i(z_t)} - y_t - u_t - c_e) \\
-\xi^{sell}_{s_{i-1}(z_t)}(z_t) - c_0 \qquad & \quad -\xi^{sell}_{s_i(z_t)}(z_t) - c_0.
\end{aligned} \tag{15}$$

Solving equation (15) for $y_t + u_t$ delivers the threshold valuation at which trader t is indifferent between the submission of a sell order at price $p^{sell}_{t,s_{i-1}(z_t)}$ and a sell order at price $p^{sell}_{t,s_i(z_t)}$

$$\begin{aligned}
&\theta^{sell}_{s_{i-1}(z_t),s_i(z_t)}(z_t) \\
&\quad = y_t + u_t \\
&\quad = p^{sell}_{t,s_{i-1}(z_t)} - c_e - \frac{\left(p^{sell}_{t,s_i(z_t)} - p^{sell}_{t,s_{i-1}(z_t)}\right)\psi^{sell}_{s_i(z_t)}(z_t) + \left(\xi^{sell}_{s_{i-1}(z_t)}(z_t) - \xi^{sell}_{s_i(z_t)}(z_t)\right)}{\psi^{sell}_{s_{i-1}(z_t)}(z_t) - \psi^{sell}_{s_i(z_t)}(z_t)}
\end{aligned} \tag{16}$$

which is a function of order prices, trader t's subjective beliefs about execution probabilities and picking off risks and the order submission and execution costs. Similarly derived, the threshold valuation associated with indifference between a sell order sub-

[13] In their empirical application Hollifield, Miller, and Sandås (2004) give evidence for the validity of the monotonicity property at least for buy and sell orders considered separately. They reject the monotonicity restrictions for buy and sell orders considered jointly and argue that this is due to inadequacies of the model to some extent.

mitted at price $p^{sell}_{t,s_i(z_t)}$ and no order submission is given by

$$\theta^{sell}_{s_i(z_t),no}(z_t) = p^{sell}_{t,s_i(z_t)} - c_e - \frac{\xi^{sell}_{s_i(z_t)}(z_t) + c_0}{\psi^{sell}_{s_i(z_t)}(z_t)}. \tag{17}$$

The threshold valuation associated with indifference between the submission of a buy order submitted at price $p^{buy}_{t,b_{i-1}(z_t)}$ and a buy order submitted at price $p^{buy}_{t,b_i(z_t)}$ looks as follows

$$\begin{aligned} &\theta^{buy}_{b_{i-1}(z_t),b_i(z_t)}(z_t) \\ &\quad = p^{buy}_{t,b_{i-1}(z_t)} + c_e + \frac{\left(p^{buy}_{t,b_{i-1}(z_t)} - p^{buy}_{t,b_i(z_t)}\right)\psi^{buy}_{b_i(z_t)}(z_t) + \left(\xi^{buy}_{b_i(z_t)}(z_t) - \xi^{buy}_{b_{i-1}(z_t)}(z_t)\right)}{\psi^{buy}_{b_{i-1}(z_t)}(z_t) - \psi^{buy}_{b_i(z_t)}(z_t)} \end{aligned} \tag{18}$$

while the threshold valuation associated with indifference between the submission of a buy order at price $p^{buy}_{t,b_i(z_t)}$ and no order submission is given by

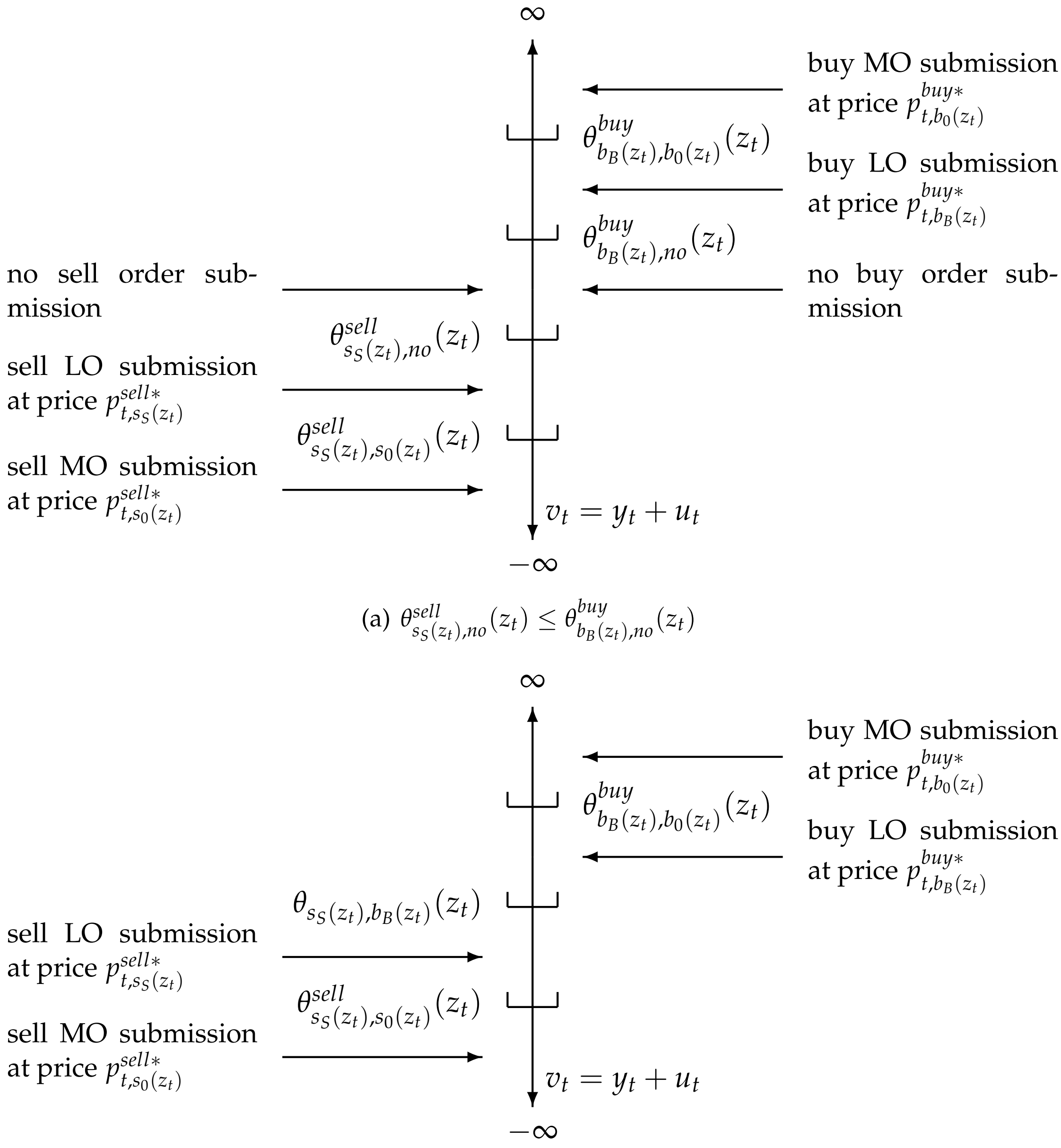
$$\theta^{buy}_{b_i(z_t),no}(z_t) = p^{buy}_{t,b_i(z_t)} + c_e - \frac{\xi^{buy}_{b_i(z_t)}(z_t) - c_0}{\psi^{buy}_{b_i(z_t)}(z_t)}. \tag{19}$$

The case of indifference between the submission of a buy order at price $p^{buy}_{t,b_i(z_t)}$ and the submission of a sell order at price $p^{sell}_{t,s_i(z_t)}$ is captured by the threshold

$$\begin{aligned} &\theta_{s_i(z_t),b_i(z_t)}(z_t) \\ &\quad = \frac{\left(p^{buy}_{t,b_i(z_t)}\psi^{buy}_{b_i(z_t)}(z_t) + p^{sell}_{t,s_i(z_t)}\psi^{sell}_{s_i(z_t)}(z_t)\right) + c_e\left(\psi^{buy}_{b_i(z_t)}(z_t) - \psi^{sell}_{s_i(z_t)}(z_t)\right)}{\psi^{sell}_{s_i(z_t)}(z_t) + \psi^{buy}_{b_i(z_t)}(z_t)} \\ &\qquad - \frac{\left(\xi^{buy}_{b_i(z_t)}(z_t) - \xi^{sell}_{s_i(z_t)}(z_t)\right)}{\psi^{sell}_{s_i(z_t)}(z_t) + \psi^{buy}_{b_i(z_t)}(z_t)}. \end{aligned} \tag{20}$$

To illustrate the link between traders' valuations for the stock and their optimal order submission strategies, consider the case for which the set of optimal order submissions is described by $\mathcal{S}^*(z_t) = \{s_0(z_t), s_S(z_t)\}$ and $\mathcal{B}^*(z_t) = \{b_0(z_t), b_B(z_t)\}$, i.e. some traders find it optimal to submit buy or sell market orders at prices $p^{buy*}_{t,b_0(z_t)}$ and $p^{sell*}_{t,s_0(z_t)}$, while others find it optimal to submit marginal buy or sell limit orders at prices $p^{buy*}_{t,b_B(z_t)}$ and $p^{sell*}_{t,s_S(z_t)}$. Under the validity of the monotonicity property the optimal order submission strategies of traders with different valuations $v_t = y_t + u_t$ for an asset illustrated in terms of threshold valuations can look as cases (a) and (b) in figure 1.

Figure 1: **Optimal order submission strategies in a LOB market**



(a) $\theta^{sell}_{s_S(z_t),no}(z_t) \leq \theta^{buy}_{b_B(z_t),no}(z_t)$

(b) $\theta^{buy}_{b_B(z_t),no}(z_t) \leq \theta_{s_S(z_t),b_B(z_t)}(z_t) \leq \theta^{sell}_{s_S(z_t),no}(z_t)$

Before shedding light on the cases' differences the focus lies on their similarities: in both cases the threshold valuations form a monotone sequence on the valuation axis $v_t = y_t + u_t$ such that order submissions are uniquely related to threshold intervals.

More precisely, the figure has to be read as follows: given a realization of the common value of the stock y_t the realization of trader t's private value of the stock u_t determines his optimal order submission strategy via the interval his valuation v_t lies in.

Consider for example the interval $[\theta^{buy}_{b_B(z_t),b_0(z_t)}(z_t), +\infty)$ that implies for a given y_t extremely high private valuations u_t, then traders with valuations v_t ranging within this interval will submit buy market orders at price $p^{buy*}_{t,b_0}(z_t)$ since they exhibit the highest execution probability. Similarly, traders with extremely low private valuations u_t exhibit valuations v_t that range within the interval $(-\infty, \theta^{sell}_{s_S(z_t),s_0(z_t)}(z_t))$ such that they submit sell market orders at price $p^{sell*}_{t,s_0}(z_t)$. Traders with intermediate private valuations u_t either submit no order or orders with low execution probabilities. At this point cases (a) and (b) need to be distinguished.

Case (a) shows traders' optimal order submission strategies in a LOB market for $\theta^{sell}_{s_S(z_t),no}(z_t) \leq \theta^{buy}_{b_B(z_t),no}(z_t)$ such that there is a range of valuations, namely $\theta^{sell}_{s_S(z_t),no}(z_t) \leq v_t < \theta^{buy}_{b_B(z_t),no}(z_t)$, within which no orders are submitted. Hence, for a given y_t traders with intermediate private valuations u_t either submit no order if their valuations v_t range within $[\theta^{sell}_{s_S(z_t),no}(z_t), \theta^{buy}_{b_B(z_t),no}(z_t))$ or they submit buy or sell limit orders with low execution probabilities if their valuations v_t range within $[\theta^{buy}_{b_B(z_t),no}(z_t), \theta^{buy}_{b_B(z_t),b_0(z_t)}(z_t))$ or $[\theta^{sell}_{s_S(z_t),s_0(z_t)}(z_t), \theta^{sell}_{s_S(z_t),no}(z_t))$. Case (b) shows traders' optimal order submission strategies in a LOB market for $\theta^{buy}_{b_B(z_t),no}(z_t) \leq \theta_{s_S(z_t),b_B(z_t)}(z_t) \leq \theta^{sell}_{s_S(z_t),no}(z_t)$ such that for any possible valuation a trader submits some order: for a given y_t traders with intermediate private valuations u_t either submit a buy limit order if their private valuations u_t lie within $[\theta_{s_S(z_t),b_B(z_t)}(z_t), \theta^{buy}_{b_B(z_t),b_0(z_t)}(z_t))$ or a sell limit order if their private valuations u_t lie within $[\theta^{sell}_{s_S(z_t),s_0(z_t)}(z_t), \theta_{s_S(z_t),b_B(z_t)}(z_t))$. To capture the distinction of optimal order submissions for intermediate private valuations, define as marginal thresholds for sellers and buyers

$$\begin{aligned} \theta^{buy}_{marginal}(z_t) &= \max(\theta_{s_S(z_t),b_B(z_t)}(z_t), \theta^{buy}_{b_B(z_t),no}(z_t)), \\ \theta^{sell}_{marginal}(z_t) &= \min(\theta_{s_S(z_t),b_B(z_t)}(z_t), \theta^{sell}_{s_S(z_t),no}(z_t)). \end{aligned} \tag{21}$$

The optimal order submission strategies in terms of threshold valuations in the general case, i.e. allowing for the full set of potential optimal order submissions $\mathcal{S}^*(z_t) = \{s_0(z_t), s_1(z_t), \ldots, s_S(z_t)\}$ and $\mathcal{B}^*(z_t) = \{b_0(z_t), b_1(z_t), \ldots, b_B(z_t)\}$, can formally be summarized as follows:

$$d_s^{sell*}(y_t + u_t; z_t) = 0, \quad \text{for } s \notin S^*(z_t) \tag{22}$$

$$d_0^{sell*}(y_t + u_t; z_t) = \begin{cases} 1, & \text{if } -\infty \leq y_t + u_t < \theta^{sell}_{s_0(z_t), s_1(z_t)}(z_t), \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

$$d_{s_i(z_t)}^{sell*}(y_t + u_t; z_t) = \begin{cases} 1, & \text{if } s_i(z_t) \notin \{0, s_S(z_t)\} \text{ and} \\ & \theta^{sell}_{s_{i-1}(z_t), s_i(z_t)}(z_t) \leq y_t + u_t < \theta^{sell}_{s_i(z_t), s_{i+1}(z_t)}(z_t), \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

$$d_{s_S(z_t)}^{sell*}(y_t + u_t; z_t) = \begin{cases} 1, & \text{if } \theta^{sell}_{s_{S-1}(z_t), s_S(z_t)}(z_t) \leq y_t + u_t < \theta^{sell}_{marginal}(z_t), \\ 0, & \text{otherwise,} \end{cases} \tag{25}$$

for the sell side of the market, with the buy side looking similarly.

3.2 Construction of Market Efficiency Measures

HMSS use as market efficiency measures the gains from trade at an order driven stock exchange, namely current gains CG, as a percentage of the theoretical benchmarks of either the maximum possible, denoted as maximum gains MaG, or monopoly-induced gains from trade, called monopoly gains MoG:

$$\text{Market efficiency measure I} = \frac{CG}{MaG} \times 100[\%], \tag{26}$$

$$\text{Market efficiency measure II} = \frac{CG}{MoG} \times 100[\%]. \tag{27}$$

To guard against confusion, what HMSS sloppily call current, maximum and monopoly gains are not the gains from trade accruing from a transaction in the respective market form as a whole, but rather quantifications of how much we expect a single trader with unknown valuation for an asset to contribute to the gains from trade in the three market forms. How the market efficiency measures' ingredients can be derived on the basis of the above described theoretical relations is presented in the following.

In the LOB market gains from trade accrue as a consequence of transactions between market and limit order traders. Consider for example a sell market order trader with private valuation $u^{sell}_{t+\tau}$ and a buy limit order trader with private valuation u^{buy}_{t} who transact at time $t+\tau$ at price $p^{buy}_{t,b} = p^{sell}_{t+\tau,0}$. Under these conditions the exchange surplus in terms of the traders' realized utilities is given by

$$\begin{aligned}(p^{sell}_{t+\tau,0} - y_{t+\tau} - u^{sell}_{t+\tau} - c_e - c_0) + (y_{t+\tau} + u^{buy}_{t} - p^{buy}_{t,b} - c_e - c_0)\\ = (-u^{sell}_{t+\tau} - c_e - c_0) + (u^{buy}_{t} - c_e - c_0), \quad (28)\end{aligned}$$

with the seller's contribution to the gains from trade equal to $-u^{sell}_{t+\tau} - c_e - c_0$ and the buyer's contribution to the gains from trade equal to $u^{buy}_{t} - c_e - c_0$. The result for a sell limit order executing with a buy market order looks similarly, whereas in the case of a non-executing limit order the limit order trader's contribution to the gains from trade is negative and given by $-c_0$.

However, what HMSS call current gains from trade realized in state z_t is not trader t's ex-post contribution to the gains from trade that accrues as a consequence of a particular trade, but his expected contribution to the gains from trade in a given state before his valuation is known. Consider a trader with unknown valuation $v_t = y_t + u_t$ arriving at state z_t, then it is the randomness of his valuation that spans a whole set of potential optimal order submissions, which in turn together with the appropriate execution probabilities span a whole set of potential contributions to the gains from trade. Taking expectations conditional on z_t over the set of potential contributions to the gains from trade delivers what is defined as current gains from trade:

$$CG(z_t) = \mathbb{E}\left[\begin{array}{c} \sum_{s=0}^{S} d^{sell*}_{s}(y_t + u_t; z_t)\left(\psi^{sell}_{s}(z_t)(-u_t - c_e) - c_0\right) \\ + \sum_{b=0}^{B} d^{buy*}_{b}(y_t + u_t; z_t)\left(\psi^{buy}_{b}(z_t)(u_t - c_e) - c_0\right) \end{array} \middle| \, z_t \right]. \quad (29)$$

Equation (29) depends on z_t but the computation of the final market efficiency measures from (26) and (27) requires an expression for the unconditional current gains from trade, i.e. a quantification of how much a single trader arriving in t with unknown valuation for an asset is expected to contribute to the gains from trade without knowing the realization of z_t, expectations across all z_t need to be taken to deliver

$$CG = \mathbb{E}\left[CG(z_t)|\text{Trader arrives and submits an order}\right], \tag{30}$$

the unconditional expected current gains from trade.

The maximum gains from trade can be achieved in a perfectly liquid market which only exists as a theoretical benchmark delivering an upper bound on the gains from trade attainable in any real world trading mechanism. In order to determine the expected contribution of a single trader with unknown valuation for the stock arriving in state x_t to the maximum gains from trade, a link between his optimal order submission strategy and his valuation for the stock in this market form is needed. To provide this link, the stock allocation is described with the help of the indicator function:

$$I^{sell}(u_t; x_t) = \begin{cases} 1, & \text{if a trader with private value } u_t \text{ sells the stock in state } x_t, \\ 0, & \text{otherwise,} \end{cases} \tag{31}$$

with the buy indicator function defined similarly. The stock allocation that delivers the maximum expected gains from trade for trader t is derived by solving the following optimization problem

$$\max_{\{I^{sell}(u_t;x_t), I^{buy}(u_t;x_t)\}} \mathbb{E}\left[I^{sell}(u_t; x_t)(-u_t - c_e - c_0) + I^{buy}(u_t; x_t)(u_t - c_e - c_0) \mid x_t\right], \tag{32}$$

subject to

$$I^{sell}(u_t; x_t) + I^{buy}(u_t; x_t) \leq 1, \quad \text{for all } u_t, \tag{33}$$

$$\mathbb{E}\left[I^{sell}(u_t; x_t) \mid x_t\right] = \mathbb{E}\left[I^{buy}(u_t; x_t) \mid x_t\right], \tag{34}$$

where (33) represents the one-shot market characteristic and (34) is the market clearing condition. Assuming the distribution of trader t's private valuation for the stock to be continuous and symmetric with median zero, then HMSS show that the optimal stock allocation of trader t is given by

$$I^{sell*}(u_t; x_t) = \begin{cases} 1, & \text{for } u_t \leq -c_e - c_0, \\ 0, & \text{otherwise,} \end{cases} \tag{35}$$

$$I^{buy*}(u_t; x_t) = \begin{cases} 1, & \text{for } u_t \geq c_e + c_0, \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

These results deliver the link between trader t's valuation for the stock and his optimal order submission strategy in the perfectly liquid market that is needed to assess trader t's expected contribution to the maximum gains from trade. As in the case of the current gains from trade, the randomness of trader t's private valuation for the stock implies that, via the randomness of the optimal order submission strategy, his contribution to the maximum gains from trade is also random. Taking expectations conditional on x_t over the potential contributions to the gains from trade delivers what is called maximum gains from trade:

$$MaG(x_t) \quad = \quad \mathbb{E}\left[\begin{array}{c} I^{sell*}(u_t; x_t)(-u_t - c_e - c_0) \\ \quad +I^{buy*}(u_t; x_t)(u_t - c_e - c_0) \end{array} \middle| \, x_t \right]. \tag{37}$$

As in the case of the current gains from trade expectations across all states x_t need to be taken to deliver

$$MaG = \mathbb{E}\left[MaG(x_t) | \text{Trader arrives and submits an order}\right], \tag{38}$$

the unconditional expected maximum gains from trade which are needed to compute the final market efficiency measure from (26).[14]

[14] In addition HMSS provide the means for a decomposition of the losses, measured as the difference between the maximum and current gains from trade, into the four sources (i) no execution, (ii) no submission, (iii) wrong direction and (iv) extramarginal submission. This master thesis abstains from doing so for shortage of space.

The monopoly gains from trade result from a market in which liquidity is supplied by a profit maximizing monopolist. In this market form traders are price takers in the sense that market prices are exogenous. In order to determine the expected contribution of a single trader with unknown valuation for the stock arriving in state x_t to the monopoly gains from trade, trader t's optimal response to the monopolist's profit maximizing quotes, the bid b_t^m and the ask a_t^m, as a function of his valuation for the stock needs to be found. To provide this link, the order submission strategy of trader t is described with the help of the following indicator function:

$$I^{m,sell}(b_t^m; u_t; x_t) = \begin{cases} 1, & \text{if a trader with private value } u_t \text{ sells the stock in state } x_t, \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

with the buy indicator function defined similarly. The monopolist determines his profit maximizing quotes by solving the following optimization problem

$$\max_{\{b_t^m, a_t^m\}} \mathbb{E}\left[I^{m,sell}(b_t^m; u_t; x_t)(y_t - b_t^m) + I^{m,buy}(a_t^m; u_t; x_t)(a_t^m - y_t)\right], \tag{40}$$

subject to

$$\mathbb{E}\left[I^{m,sell}(b_t^m; u_t; x_t) \mid x_t\right] = \mathbb{E}\left[I^{m,buy}(a_t^m; u_t; x_t) \mid x_t\right], \tag{41}$$

where (41) is the market clearing condition. Assuming the distribution of trader t's private valuation for the stock to be continuous and symmetric with median zero, then HMSS show that the monopolist's profit maximizing quotes are given by

$$b_t^{m*} = y_t - \frac{G(b_t^{m*} - c_e - c_0 - y_t|x_t)}{g(b_t^{m*} - c_e - c_0 - y_t|x_t)}, \tag{42}$$

$$a_t^{m*} = y_t + \frac{1 - G(a_t^{m*} + c_e + c_0 - y_t|x_t)}{g(a_t^{m*} + c_e + c_0 - y_t|x_t)}, \tag{43}$$

while trader t's optimal order submission strategy is described by

$$I^{m,sell}(b_t^{m*}; u_t; x_t) = \begin{cases} 1, & \text{for } u \leq b_t^{m*} - c_e - c_0 - y_t \\ 0, & \text{otherwise,} \end{cases} \tag{44}$$

$$I^{m,buy}(a_t^{m*}; u_t; x_t) = \begin{cases} 1, & \text{for } u \geq a_t^{m*} + c_e + c_0 - y_t \\ 0, & \text{otherwise.} \end{cases} \tag{45}$$

These results deliver trader t's optimal response to the monopolist's profit maximizing quotes as a function of his private valuation for the stock which is needed to assess trader t's expected contribution to the monopoly gains from trade. Once again it is the randomness of trader t's private valuation for the stock that implies not only the randomness of the optimal order submission strategy, but also the fact that trader t's contribution to the monopoly gains from trade is random. Taking expectations conditional on x_t over the potential contributions to the gains from trade delivers what is called monopoly gains from trade:

$$MoG(x_t) = \mathbb{E}\left[\begin{array}{c} I^{m,sell}(b_t^{m*}; u_t; x_t)(-u_t - c_e - c_0) \\ +I^{m,buy}(a_t^{m*}; u_t; x_t)(u_t - c_e - c_0) \end{array} \middle| x_t \right]. \tag{46}$$

Once again, as for the current and the maximum gains from trade, expectations across all states x_t need to be taken to deliver

$$MoG = \mathbb{E}\left[MoG(x_t)|\text{Trader arrives and submits an order}\right], \tag{47}$$

the unconditional expected monopoly gains from trade which are needed to compute the final market efficiency measure from (27).

In the sense that the hypothetical benchmark of the maximum gains from trade provides an upper bound on the gains from trade attainable in any feasible mechanism, the monopoly gains from trade may not exactly match the concept of minimum possible gains from trade that are conceivable theoretically. Nevertheless, they provide a lower bound on the gains from trade attainable in any feasible mechanism since a monopoly market is by far the most unprofitable environment for agents to trade in reality.

With the ingredients derived, the market efficiency measures as introduced above can be constructed at least theoretically. How numbers can be attached to these theoretical constructs is subject of the next section.

4 Recipe for Estimating the Gains from Trade in Xetra LOB Market

4.1 Econometric Implementation

This paragraph deals with the econometric implementation of the theoretical model in order to provide closed form solutions of the above presented theoretical market efficiency measures. In the course of this paragraph first the two step estimation procedure proposed by HMSS will be presented in detail as to provide a structured recipe for the planned application to the Xetra trading system. Second the data requirements to implement the estimation procedure will be summarized.

First Step Estimation

On the one hand the first step estimation comprises the maximum likelihood estimation of a competing risks model for the latent cancelation and execution times.[15] On the other hand it implies the ordinary least squares estimation of regression models for the expected common value changes conditional on order execution. The former delivers formulations of the distribution functions of latent cancelation and execution times that allow to compute estimates of the execution probabilities of distinct order types. The latter makes possible to compute estimates of the common value changes conditional on order execution. Both the estimates of the execution probabilities and the estimates of the common value changes are used to compute estimates of the respective picking

[15] For a brief summary of competing risks model theory essentials see appendix A.1.

off risks. The procedure of the first step estimation is illustrated in figure 2 with the steps 1.1 through 1.5 described in detail subsequently.[16]

1.1 MLE of competing risks models for latent cancelation and execution times

In the present application the cancelation and the execution of an order constitute two reasons for an order to quit the LOB. In competing risks models' terminology these reasons are called failure types which occur at the random failure times

$$t + \tau_{cancel} \quad : \quad \text{latent cancelation time,} \tag{48}$$

$$t + \tau_{execute} \quad : \quad \text{latent execution time.} \tag{49}$$

Core of the maximum likelihood estimations of the competing risks models for the latent cancelation and execution times are the corresponding log-likelihood functions as given for a general competing risks model in equation (96) in appendix A.1.1. The log-likelihood functions' ingredients, i.e. the cancelation and execution hazard rates compare their general formulation in equation (98) in appendix A.1.1, are computed for the sets of conditioning information $\{z_t, d_{t,1}^{buy} = 1\}$, $\{z_t, d_{t,1}^{sell} = 1\}$, $\{z_t, d_{t,B(z_t)}^{buy} = 1\}$ and $\{z_t, d_{t,S(z_t)}^{sell} = 1\}$ in this application.

For the purpose of illustration, the computation of the hazard rates and the formation of the log-likelihood function are presented for the conditioning information set $\{z_t, d_{t,1}^{buy} = 1\}$, i.e. conditional on z_t and the submission of a one-tick buy limit order. The latent execution and cancelation times of this order type are assumed to follow independent conditional Weibull distributions that look as in equations (50) and (51):

$$F_{cancel}(\tau | z_t, d_{t,1}^{buy} = 1) = 1 - \exp\left(- \exp(z_t' \gamma_1^{buy}) \tau^{\alpha_1^{buy}} \right) \tag{50}$$

[16]The parameter estimates required for the computation of both the estimates of the execution probabilities and the estimates of the picking off risks are derived for the one-tick and the marginal buy and sell limit orders (that execute) only, constituting a subset of the event data set indexed by $j = 1, \dots, J$ ($l = 1, \dots, L$). These are sufficient to derive upper and lower bounds of the current gains from trade as well as the average current gains from trade for each z_{t_i} of the full sample indexed by $i = 1, \dots, I$. For further explanations see section 4.2.

Figure 2: **Recipe for first step estimation**

1.1

MLE of competing risks models for latent cancelation and execution times

Data requirements:

$[t_j + \tau_j, z_{t_j}, I_{t_j}(\tau_{execute} > \tau_{cancel}), I_{t_j}(\tau_{execute} \leq \tau_{cancel}), d^{buy}_{t_j,1}, d^{sell}_{t_j,1}, d^{buy}_{t_j,B(z_{t_j})}, d^{sell}_{t_j,S(z_{t_j})}]$ for all events $j = 1, \ldots, J$ associated with one-tick and marginal buy and sell LOs

Estimates:

- $\hat{\alpha}^{buy}_1, \hat{\beta}^{buy}_1, \hat{\gamma}^{buy}_1, \hat{\kappa}^{buy}_1$
- $\hat{\alpha}^{sell}_1, \hat{\beta}^{sell}_1, \hat{\gamma}^{sell}_1, \hat{\kappa}^{sell}_1$
- $\hat{\alpha}^{buy}_{marginal}, \hat{\beta}^{buy}_{marginal}, \hat{\gamma}^{buy}_{marginal}, \hat{\kappa}^{buy}_{marginal}$
- $\hat{\alpha}^{sell}_{marginal}, \hat{\beta}^{sell}_{marginal}, \hat{\gamma}^{sell}_{marginal}, \hat{\kappa}^{sell}_{marginal}$

1.2

Computation of estimates of execution probabilities

Data requirements:

$[t_i + \tau_i, z_{t_i}, I_{t_i}(\tau_{execute} > \tau_{cancel}), I_{t_i}(\tau_{execute} \leq \tau_{cancel}), d^{buy}_{t_i,1}, d^{sell}_{t_i,1}, d^{buy}_{t_i,B(z_{t_i})}, d^{sell}_{t_i,S(z_{t_i})}]$ for all events $i = 1, \ldots, I$ within the sample, estimates from 1.1

Estimates:

- $\hat{\psi}^{buy}_1(z_{t_i})$
- $\hat{\psi}^{sell}_1(z_{t_i})$
- $\hat{\psi}^{buy}_{B(z_{t_i})}(z_{t_i})$
- $\hat{\psi}^{sell}_{S(z_{t_i})}(z_{t_i})$

1.3

OLS of regression models for common value changes

Data requirements:

$[y_{t_l}, y_{t_l+\tau_l}, z_{t_l}, I_{t_l}(\tau_{execute} \leq \tau_{cancel}) = 1, d^{buy}_{t_l,1}, d^{sell}_{t_l,1}, d^{buy}_{t_l,B(z_{t_l})}, d^{sell}_{t_l,S(z_{t_l})}]$ for all events $l = 1, \ldots, L$ associated with one-tick and marginal buy and sell LOs that execute

Estimates:

- $\hat{\Lambda}^{buy}_1$
- $\hat{\Lambda}^{sell}_1$
- $\hat{\Lambda}^{buy}_{marginal}$
- $\hat{\Lambda}^{sell}_{marginal}$

Figure 2: **Recipe for first step estimation *(continued)***

1.4

Computation of estimates of common value changes

Data requirements:

$[z_{t_i}]$ for all events $i = 1, \ldots, I$ within the sample, estimates from 1.3

Estimates:

- $\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|\ldots, d^{buy}_{t_i,1} = 1\right]$
- $\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|\ldots, d^{sell}_{t_i,1} = 1\right]$
- $\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|\ldots, d^{buy}_{t_i,B(z_{t_i})} = 1\right]$
- $\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|\ldots, d^{sell}_{t_i,S(z_{t_i})} = 1\right]$

1.5

Computation of estimates of picking off risks

Data requirements:

estimates from 1.2 and 1.4

Estimates:

- $\hat{\xi}^{buy}_1(z_{t_i}) = \hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|\ldots, d^{buy}_{t_i,1} = 1\right] \hat{\psi}^{buy}_1(z_{t_i})$
- $\hat{\xi}^{sell}_1(z_{t_i}) = \hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|\ldots, d^{sell}_{t_i,1} = 1\right] \hat{\psi}^{sell}_1(z_{t_i})$
- $\hat{\xi}^{buy}_{B(z_{t_i})}(z_{t_i}) = \hat{\mathbb{E}}\left[(y_{t_i+\tau_j} - y_{t_j})|\ldots, d^{buy}_{t_i,B(z_{t_i})} = 1\right] \hat{\psi}^{buy}_{B(z_{t_i})}(z_{t_i})$
- $\hat{\xi}^{sell}_{S(z_{t_i})}(z_{t_i}) = \hat{\mathbb{E}}\left[(y_{t_i+\tau_j} - y_{t_j})|\ldots, d^{sell}_{t_i,S(z_{t_i})} = 1\right] \hat{\psi}^{sell}_{S(z_{t_i})}(z_{t_i})$

$$F_{execute}(\tau|z_t, d_{t,1}^{buy} = 1) = 1 - \exp\left(-\exp(z_t'\kappa_1^{buy})\tau^{\beta_1^{buy}}\right) \tag{51}$$

with densities as given in equations (52) and (53):

$$f_{cancel}(\tau|z_t, d_{t,1}^{buy} = 1) = \exp\left(-\exp(z_t'\gamma_1^{buy})\tau^{\alpha_1^{buy}}\right)\left(\exp(z_t'\gamma_1^{buy})\alpha_1^{buy}\tau^{\alpha_1^{buy}-1}\right), \tag{52}$$

$$f_{execute}(\tau|z_t, d_{t,1}^{buy} = 1) = \exp\left(-\exp(z_t'\kappa_1^{buy})\tau^{\beta_1^{buy}}\right)\left(\exp(z_t'\kappa_1^{buy})\beta_1^{buy}\tau^{\beta_1^{buy}-1}\right), \tag{53}$$

where γ_1^{buy} and κ_1^{buy} are coefficient vectors of the same dimension as the state vector z_t that measure the effects of the state variables on the corresponding hazard rates. α_1^{buy} and β_1^{buy} are scalar Weibull shape parameters and can be interpreted as follows: if $\alpha_1^{buy}(\beta_1^{buy}) = 1$, then the corresponding hazard rate is independent of τ. If $\alpha_1^{buy}(\beta_1^{buy}) < 1$, then the corresponding hazard rate is decreasing in τ. If $\alpha_1^{buy}(\beta_1^{buy}) > 1$, then the corresponding hazard rate is increasing in τ.

The hazard rates for the latent cancelation and execution times for the exemplary one-tick buy limit order can be computed as shown in the following:

$$\begin{aligned} h_{cancel}(t+\tau; z_t, d_{t,1}^{buy} = 1) &= \frac{f_{cancel}(\tau|z_t, d_{t,1}^{buy} = 1)}{1 - F_{cancel}(\tau|z_t, d_{t,1}^{buy} = 1)} \\ &= \frac{\exp\left(-\exp(z_t'\gamma_1^{buy})\tau^{\alpha_1^{buy}}\right)\left(\exp(z_t'\gamma_1^{buy})\alpha_1^{buy}\tau^{\alpha_1^{buy}-1}\right)}{1 - \left(1 - \exp\left(-\exp(z_t'\gamma_1^{buy})\tau^{\alpha_1^{buy}}\right)\right)} \\ &= \exp(z_t'\gamma_1^{buy})\alpha_1^{buy}\tau^{\alpha_1^{buy}-1}, \end{aligned} \tag{54}$$

$$\begin{aligned} h_{execute}(t+\tau; z_t, d_{t,1}^{buy} = 1) &= \frac{f_{execute}(\tau|z_t, d_{t,1}^{buy} = 1)}{1 - F_{execute}(\tau|z_t, d_{t,1}^{buy} = 1)} \\ &= \frac{\exp\left(-\exp(z_t'\kappa_1^{buy})\tau^{\beta_1^{buy}}\right)\left(\exp(z_t'\kappa_1^{buy})\beta_1^{buy}\tau^{\beta_1^{buy}-1}\right)}{1 - \left(1 - \exp\left(-\exp(z_t'\kappa_1^{buy})\tau^{\beta_1^{buy}}\right)\right)} \\ &= \exp(z_t'\kappa_1^{buy})\beta_1^{buy}\tau^{\beta_1^{buy}-1}. \end{aligned} \tag{55}$$

To construct the log-likelihood function suppose J orders indexed by $j = 1, \ldots, J$ give rise to data $[t_j + \tau_j, I_{t_j}(\tau_{execute} > \tau_{cancel}), I_{t_j}(\tau_{execute} \leq \tau_{cancel}), z_{t_j}, d^{buy}_{t_j,1} = 1]$ containing all one-tick buy limit orders of the sample, where $t_j + \tau_j$ is the observed failure time, $I_{t_j}(\tau_{execute} > \tau_{cancel})$ is an indicator function for the cancelation of the jth one-tick buy limit order, $I_{t_j}(\tau_{execute} \leq \tau_{cancel})$ is an indicator function for the execution of the jth one-tick buy limit order and $\{z_{t_j}, d^{buy}_{t_j,1} = 1\}$ is the associated conditioning information.[17] [18] Then with equation (96) in appendix A.1.1 the conditional log-likelihood function used to estimate the probability distributions of the latent cancelation and latent execution times of the exemplary one-tick buy limit orders is given by

$$\begin{aligned} & logL(. \mid z_{t_j}, d^{buy}_{t_j,1} = 1) \\ &= \sum_{j=1}^{J} \Big\{ I_{t_j}(\tau_{execute} > \tau_{cancel}) \ln \Big(h_{cancel}(t_j + \tau_j; z_{t_j}, d^{buy}_{t_j,1} = 1) \Big) \\ &\quad + I_{t_j}(\tau_{execute} \leq \tau_{cancel}) \ln \Big(h_{execute}(t_j + \tau_j; z_{t_j}, d^{buy}_{t_j,1} = 1) \Big) \\ &\quad - \int_0^{t_j+\tau_j} \Big(h_{cancel}(s; z_{t_j}, d^{buy}_{t_j,1} = 1) + h_{execute}(s; z_{t_j}, d^{buy}_{t_j,1} = 1) \Big) \, ds \Big\} . \end{aligned} \tag{57}$$

In order to provide estimates of the Weibull parameters α^{buy}_1, β^{buy}_1, γ^{buy}_1 and κ^{buy}_1, equation (57) is estimated by maximum likelihood for one-tick buy limit orders treating orders that last longer than the prespecified time span ΔT as censored observations.

The hazard rates and the log-likelihood functions on the basis of the conditioning information sets $\{z_t, d^{sell}_{t,1} = 1\}$, $\{z_t, d^{buy}_{t,B(z_t)} = 1\}$ and $\{z_t, d^{sell}_{t,S(z_t)} = 1\}$ can be derived equivalently. The remaining parameters to be estimated are α^{sell}_1, β^{sell}_1, γ^{sell}_1, κ^{sell}_1, $\alpha^{buy}_{marginal}$, $\beta^{buy}_{marginal}$, $\gamma^{buy}_{marginal}$, $\kappa^{buy}_{marginal}$, $\alpha^{sell}_{marginal}$, $\beta^{sell}_{marginal}$, $\gamma^{sell}_{marginal}$ and $\kappa^{sell}_{marginal}$.

[17] $I_{t_j}(\tau_{execute} \leq \tau_{cancel})$ is defined in (4), while $I_t(\tau_{execute} > \tau_{cancel})$ is given by

$$I_t(\tau_{execute} > \tau_{cancel}) = \begin{cases} 1, & \text{if } t + \tau_{execute} > t + \tau_{cancel}, \\ 0, & \text{otherwise.} \end{cases} \tag{56}$$

[18] Note that here the notation is simplified in that the index j is used for the subset of one-tick buy limit orders, but lateron also for the subsets of one-tick sell, marginal buy and marginal sell limit orders to avoid more complex and confusing indices.

1.2 Computation of estimates of execution probabilities

Pretending to know neither the failure time nor the failure type of an order submitted at t, i.e. treating the execution of an order as random variable, then the execution probabilities of all order submissions of interest can be obtained by taking expectations of the random order execution as illustrated for the exemplary one-tick buy limit order:

$$\begin{aligned}
& \mathbb{E}\left[I_t(\tau_{execute} \leq \tau_{cancel})|z_t, d_{t,1}^{buy} = 1\right] \\
= & \int_{-\infty}^{+\infty} I_t(\tau_{execute} \leq \tau_{cancel}) f_{I_t(\tau_{execute} \leq \tau_{cancel})|z_t, d_{t,1}^{buy}=1} d\tau \\
= & \int_{0}^{\Delta T} I_t(\tau_{execute} \leq \tau_{cancel}) f_{I_t(\tau_{execute} \leq \tau_{cancel})|z_t, d_{t,1}^{buy}=1} d\tau \\
= & \int_{0}^{\Delta T} 1 \cdot f_{I_t(\tau_{execute} \leq \tau_{cancel})|z_t, d_{t,1}^{buy}=1} d\tau \\
= & \int_{0}^{\Delta T} \left(1 - F_{cancel}(\tau|z_t, d_{t,1}^{buy} = 1)\right) \cdot \left(dF_{execute}(\tau|z_t, d_{t,1}^{buy} = 1)\right) \qquad (58)
\end{aligned}$$

The second line provides the definition of the conditional expectation of a continuous random variable, while line three results as a consequence of the failure time to be bounded from below by 0 and from above by ΔT. In line four the indicator function is replaced by the value 1 standing for order execution while the 0 for otherwise can be neglected. Line five replaces the joint probability that the order has not yet been canceled by $t + \tau$ and executes between $t + \tau$ and $t + \tau + \Delta\tau$, $f_{I_t(\tau_{execute} \leq \tau_{cancel})|z_t, d_{t,1}^{buy}=1} d\tau$, by the product of the marginal probabilities $\left(1 - F_{cancel}(\tau|z_t, d_{t,1}^{buy} = 1)\right) \cdot dF_{execute}(\tau|z_t, d_{t,1}^{buy} = 1)$, where the first element is the probability that the order has not yet been canceled by $t + \tau$ while the second denotes the probability that the order executes between $t + \tau$ and $t + \tau + \Delta\tau$.[19] Taking the expectations for the one-tick sell limit order and the marginal buy and sell limit orders can be conducted similarly.

The estimates of the execution probabilities of the one-tick and the marginal buy and sell limit orders are obtained by replacing the theoretical distribution functions as illustrated for the exemplary one-tick buy limit order in (58) by their respective Weibull specifications evaluated at the parameter estimates derived in 1.1. The resulting execu-

[19] This holds because of assuming independence between the latent execution and cancelation times.

tion probability estimates of the one-tick buy limit order

$$\hat{\psi}_1^{buy}(z_{t_i}) = \int_0^{\Delta T} \exp\left(-\exp(z'_{t_i}\hat{\gamma}_1^{buy})\tau_i^{\hat{\alpha}_1^{buy}}\right) \cdot \exp\left(-\exp(z'_{t_i}\hat{\kappa}_1^{buy})\tau_i^{\hat{\beta}_1^{buy}}\right)\left(\exp(z'_{t_i}\hat{\kappa}_1^{buy})\hat{\beta}_1^{buy}\tau_i^{\hat{\beta}_1^{buy}-1}\right) d\tau_i, \quad (59)$$

and those of the one-tick sell and the marginal buy and sell limit orders $\hat{\psi}_1^{sell}(z_{t_i})$, $\hat{\psi}_{marginal}^{buy}$ (z_{t_i}) and $\hat{\psi}_1^{sell}(z_{t_i})$ are each computed for all order submissions in the sample indexed by $i = 1, \dots, I$, no matter if submitted as one-tick or marginal buy or sell limit order or not.

1.3 OLS of regression models for common value changes

The expected common value changes conditional on order execution are parameterized as linear regression models. Suppose L orders indexed by $l = 1, \dots, L$ give rise to data $[y_{t_l}, y_{t+\tau_{executel}}, z_{t_l}, I_{t_l}(\tau_{execute} \leq \tau_{cancel}) = 1, d_{t_l,1}^{buy}, d_{t_l,1}^{sell}, d_{t_l,B(z_{t_l})}^{buy}, d_{t_l,S(z_{t_l})}^{sell}]$ containing one-tick and marginal buy and sell limit orders that execute, then the four regression models

$$\mathbb{E}\left[(y_{t+\tau_{executel}} - y_{t_l})|I_{t_l}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_l}, d_{t_l,1}^{buy} = 1\right] = z'_{t_l}\Lambda_1^{buy} \quad (60)$$

$$\mathbb{E}\left[(y_{t+\tau_{executel}} - y_{t_l})|I_{t_l}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_l}, d_{t_l,1}^{sell} = 1\right] = z'_{t_l}\Lambda_1^{sell} \quad (61)$$

$$\mathbb{E}\left[(y_{t+\tau_{executel}} - y_{t_l})|I_{t_l}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_l}, d_{t_l,B(z_t)}^{buy} = 1\right] = z'_{t_l}\Lambda_{marginal}^{buy} \quad (62)$$

$$\mathbb{E}\left[(y_{t+\tau_{executel}} - y_{t_l})|I_{t_l}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_l}, d_{t_l,S(z_t)}^{sell} = 1\right] = z'_{t_l}\Lambda_{marginal}^{sell} \quad (63)$$

can be estimated by OLS to provide estimates of Λ_1^{buy}, Λ_1^{sell}, $\Lambda_{marginal}^{buy}$ and $\Lambda_{marginal}^{sell}$.

1.4 Computation of estimates of common value changes

Suppose I orders indexed by $i = 1, \dots, I$ give rise to data $[z_{t_i}]$ containing the state vectors of all order submissions in the sample, then estimates of the expected changes in the common value at every order submission are delivered for the exemplary one-tick buy limit order by computing

$$\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|I_{t_i}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_i}, d_{t_i,1}^{buy} = 1\right] = z'_{t_i}\hat{\Lambda}_1^{buy}. \quad (64)$$

The estimates of the common value changes for the sell one-tick and the buy and sell marginal limit orders at every order submission, i.e. $\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|I_{t_i}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_i}, d^{sell}_{t_i,1} = 1\right]$, $\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|I_{t_i}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_i}, d^{buy}_{t_i,B(z_t)} = 1\right]$ and $\hat{\mathbb{E}}\left[(y_{t_i+\tau_i} - y_{t_i})|I_{t_i}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_i}, d^{sell}_{t_i,S(z_t)} = 1\right]$, can be computed similarly.

1.5 Computation of estimates of picking off risks

In order to derive estimates of the picking off risks of the exemplary one-tick buy limit order at every order submission in the sample indexed by $i = 1, \ldots, I$, substituting the appropriate estimates of the execution probabilities and the associated estimates of the common-value changes into equation (9) delivers:

$$\hat{\xi}^{buy}_1(z_{t_i}) = \hat{\mathbb{E}}\left[(y_{t+\tau_{execute i}} - y_{t_i}) \mid I_{t_i}(\tau_{execute} \leq \tau_{cancel}) = 1, z_{t_i}, d^{buy}_{t_i,1} = 1\right] \hat{\psi}^{buy}_1(z_{t_i}). \quad (65)$$

Similarly, the estimates of the picking off risks of the one-tick sell and the marginal buy and sell limit orders at every order submission, i.e. $\hat{\xi}^{sell}_1(z_{t_i})$, $\hat{\xi}^{buy}_{B(z_{t_i})}(z_{t_i})$ and $\hat{\xi}^{sell}_{S(z_{t_i})}(z_{t_i})$, are obtained.

Second Step Estimation

The second step estimation deals with the maximum likelihood estimation of a competing risks model for the timing of market and limit orders that delivers the remaining parameters needed to actually calculate the market efficiency measures from (26) and (27). The results of the first step estimation enter the likelihood function of the second step estimation via the threshold functions that characterize the optimal order submission strategies. The estimation of the likelihood function further requires parameterizations of the trader arrival rates and the private value distribution. The procedure of the second step estimation is illustrated in figure 3 with the steps 2.1 through 2.4 to be handled described in a detailed recipe provided in the text.

Figure 3: **Recipe for second step estimation**

2.1

Weibull model for trader arrival rate

Model:

$\lambda(t; x_{t_i})dt =$
$\exp(x'_{t_i}\delta)\eta(t - t_{i-1})^{\eta-1}dt$

Parameters:

- δ
- η

2.2

Mixture of two normal distributions for private value distribution

Model:

$G(u|x_t) =$
$\rho\Phi\left(\frac{u}{\sigma_1^*(y_t,x_t)}\right) + (1-\rho)\Phi\left(\frac{u}{\sigma_2^*(y_t,x_t)}\right)$

with $\sigma_1^*(y_t, x_t) = y_t\sigma_1 \exp(x_t'\Gamma)$
and $\sigma_2^*(y_t, x_t) = y_t\sigma_2 \exp(x_t'\Gamma)$

Parameters:

- σ_1
- σ_2
- ρ
- Γ

2.3

Evaluation of threshold functions at first step estimates

Threshold functions:

$\theta^{buy}_{1,0}(z_{t_i})$, $\theta^{buy}_{marginal}(z_{t_i})$, $\theta^{sell}_{marginal}(z_{t_i})$, $\theta^{sell}_{1,0}(z_{t_i})$ eval. at $\hat{\psi}^{buy}_1(z_{t_i})$, $\hat{\psi}^{sell}_1(z_{t_i})$, $\hat{\psi}^{buy}_{B(z_{t_i})}(z_{t_i})$, $\hat{\psi}^{sell}_{S(z_{t_i})}(z_{t_i})$, $\hat{\xi}^{buy}_1(z_{t_i})$, $\hat{\xi}^{sell}_1(z_{t_i})$, $\hat{\xi}^{buy}_{B(z_{t_i})}(z_{t_i})$, $\hat{\xi}^{sell}_{S(z_{t_i})}(z_{t_i})$ from first step estimation

Parameters:

- c_0
- c_e

2.4

MLE of competing risks model for timing of market and limit orders

Data requirements:

$[t_i, d^{sell}_{t_i,s}, d^{buy}_{t_i,b}, x_{t_i}]$ for $s = 0, 1, \ldots, S(z_{t_i})$ and $b = 0, 1, \ldots, B(z_{t_i})$ for all events $i = 1, \ldots, I$ in the sample, parameterizations from 2.1 and 2.2 as well as threshold functions evaluated at parameter estimates from 2.3

Estimates:

- $\hat{\delta}$, $\hat{\eta}$
- $\hat{\sigma}_1$, $\hat{\sigma}_2$, $\hat{\rho}$, $\hat{\Gamma}$
- $\hat{c}_0$, $\hat{c}_e$

2.1 Weibull model for trader arrival rate

The trader arrival rate from equation (1) is parameterized as the following Weibull model

$$\lambda(t; x_{t_i})dt = \exp(x'_{t_i}\delta)\eta(t - t_{i-1})^{\eta-1}dt, \tag{66}$$

which supposes that the last order submission was at t_{i-1}. δ is a coefficient vector of the same dimension as the exogenous state variables x_{t_i} and measures the effects of the latter on the hazard rate $\lambda(t; x_{t_i})$. The scalar Weibull shape parameter η is interpreted as in the case of the Weibull formulations for the latent cancelation and execution times.

2.2 Mixture of two normal distributions for private value distribution

The conditional private value distribution from equation (3) is parameterized as a mixture of two normal distributions with mean zero and standard deviations as functions of the common value and the exogenous state variables

$$\sigma_1^*(y_t, x_t) = y_t\sigma_1 \exp(x'_t\Gamma), \tag{67}$$

$$\sigma_2^*(y_t, x_t) = y_t\sigma_2 \exp(x'_t\Gamma), \tag{68}$$

where $\sigma_1 \neq \sigma_2$, and looks as follows:

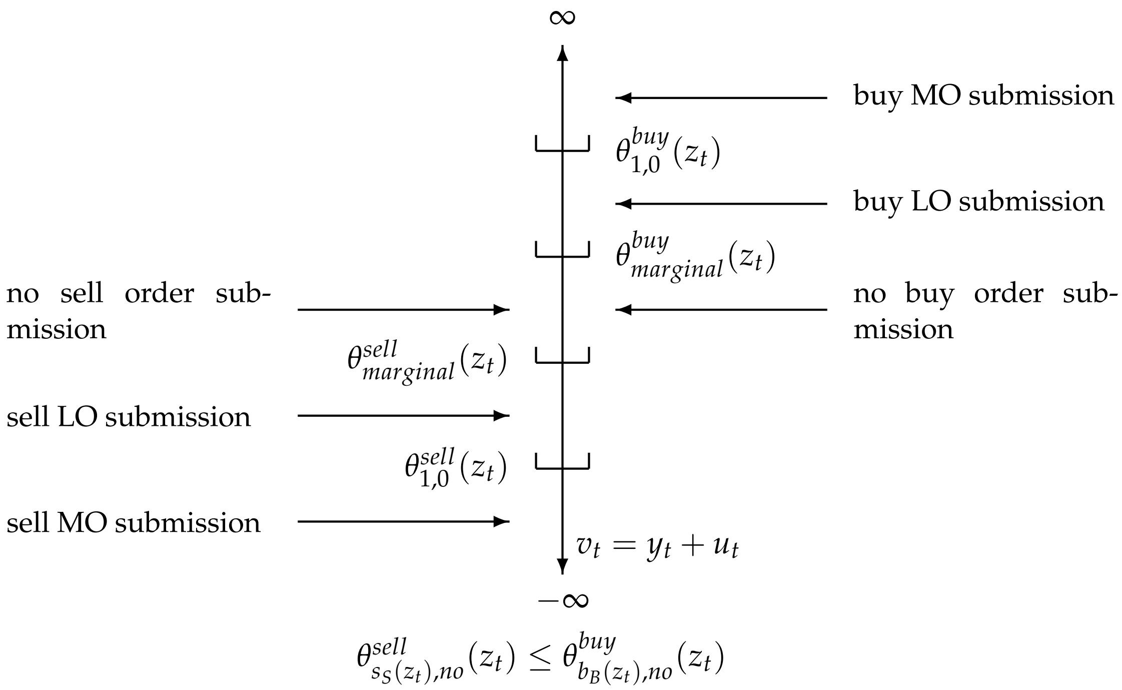
$$G(u|x_t) = \rho\Phi\left(\frac{u}{\sigma_1^*(y_t, x_t)}\right) + (1-\rho)\Phi\left(\frac{u}{\sigma_2^*(y_t, x_t)}\right). \tag{69}$$

The symbol Φ stands for the normal cumulative distribution function, ρ satisfying $0 < \rho < 1$ denotes the weighting factor, while Γ is a coefficient vector of the same dimension as the exogenous state vector x_t that measures the effect of the latter on the standard deviations $\sigma_1^*(y_t, x_t)$ and $\sigma_2^*(y_t, x_t)$.

2.3 Evaluation of threshold functions at first step estimates

Under the assumption that the log-likelihood function for the timing of market and limit orders will be formed for buy and sell market orders and buy and sell limit orders between the one-tick and the marginal limit orders, the threshold valuations needed for the computation of the log-likelihood function are those illustrated in figure 4.[20]

Figure 4: **Optimal order submission strategies for MLE of logL$(. \mid \mathbf{z_t})$**



The first step estimates of the execution probabilities $\hat{\psi}^{buy}_1(z_{t_i})$, $\hat{\psi}^{sell}_1(z_{t_i})$, $\hat{\psi}^{buy}_{B(z_{t_i})}(z_{t_i})$ and $\hat{\psi}^{sell}_{S(z_{t_i})}(z_{t_i})$ for all order submissions in the sample indexed by $i = 1, \ldots, I$ and the associated first step estimates of the picking off risks $\hat{\xi}^{buy}_1(z_{t_i})$, $\hat{\xi}^{sell}_1(z_{t_i})$, $\hat{\xi}^{buy}_{B(z_{t_i})}(z_{t_i})$ and $\hat{\xi}^{sell}_{S(z_{t_i})}(z_{t_i})$ are substituted into the threshold functions from equations (16) to (20) in order to deliver formulations of the threshold valuations at every order submission $\theta^{buy}_{1,0}(z_{t_i})$, $\theta^{buy}_{marginal}(z_{t_i})$, $\theta^{sell}_{marginal}(z_{t_i})$ and $\theta^{sell}_{1,0}(z_{t_i})$ leaving solely the parameters c_0 and c_e to be estimated in order to actually compute the respective threshold valuations at every order submission.

2.4 MLE of competing risks model for timing of market and limit orders

In the present application the submissions of buy and sell market orders and buy and sell limit orders constitute four reasons (failure types) for a trader to arrive and submit

[20]Figure 4 illustrates the optimal order submission strategies under validity of the monotonicity property (see above).

an order. In order to set up this competing risks model's likelihood function, the hazard rates associated with these order submissions need to be computed.

Given the conditional private value distribution from equation (3) and the trader arrival rate from equation (1), the conditional probability of the submission of a sell market order between t and $t + dt$ is given by

$$\begin{aligned}
&\Pr(\text{Sell MO in } [t, t + dt]|z_t) \\
&= \Pr(y_t + u_t < \theta_{0,1}^{sell}) \cdot \Pr(\text{Trader arrives in } [t, t + dt) \mid x_t) \\
&= G(\theta_{0,1}^{sell}(z_t) - y_t|x_t) \cdot \lambda(t; x_t)dt,
\end{aligned} \tag{70}$$

such that the associated hazard rate is equal to

$$G(\theta_{0,1}^{sell}(z_t) - y_t|x_t)\lambda(t; x_t). \tag{71}$$

The conditional probability of the submission of a sell limit order between the one-tick and the marginal sell limit order between t and $t + dt$ is given by

$$\begin{aligned}
&\Pr(\text{Sell LO in } [t, t + dt]|z_t) \\
&= \Pr(\theta_{0,1}^{sell} \leq y_t + u_t < \theta_{marginal}^{sell}) \cdot \Pr(\text{Trader arrives in } [t, t + dt) \mid x_t) \\
&= \left[G(\theta_{marginal}^{sell}(z_t) - y_t|x_t) - G(\theta_{0,1}^{sell}(z_t) - y_t|x_t)\right] \cdot \lambda(t; x_t)dt,
\end{aligned} \tag{72}$$

such that the associated hazard rate is equal to

$$\left[G(\theta_{marginal}^{sell}(z_t) - y_t|x_t) - G(\theta_{0,1}^{sell}(z_t) - y_t|x_t)\right] \lambda(t; x_t). \tag{73}$$

Similarly derived, the hazard rate for the submission of a buy market order submitted in t equals

$$\left[1 - G(\theta_{0,1}^{buy}(z_t) - y_t|x_t)\right] \lambda(t; x_t), \tag{74}$$

while the hazard rate for a buy limit order between the one-tick and the marginal buy limit order in t is given by

$$\left[G(\theta_{0,1}^{buy}(z_t) - y_t|x_t) - G(\theta_{marginal}^{buy}(z_t) - y_t|x_t)\right] \lambda(t; x_t). \tag{75}$$

The hazard rate for the submission of an order of any type is given by the sum of equations (71), (73), (74) and (75) and equals:

$$\left[1 - G(\theta_{marginal}^{buy}(z_t) - y_t|x_t) + G(\theta_{marginal}^{sell}(z_t) - y_t|x_t)\right] \lambda(t; x_t). \tag{76}$$

Suppose I orders indexed by $i = 1, \ldots, I$ give rise to data $[t_i, d_{t_i,s}^{sell}, d_{t_i,b}^{buy}, x_{t_i}]$ for $s = 0, 1, \ldots, S(z_{t_i})$ and $b = 0, 1, \ldots, B(z_{t_i})$ containing all buy and sell market and limit orders, where t_i is the observed order submission time, $d_{t_i,s}^{sell}$ is an indicator function for the ith order being a sell order, $d_{t_i,b}^{buy}$ is an indicator function for the ith order being a buy order and x_{t_i} is the associated conditioning information. Then with equation (96) in appendix A.1.1, the conditional log-likelihood function used to estimate the probability distributions for the timing of market and limit orders results as

$$\begin{aligned}
& logL(.|z_{t_i}) \\
&= \sum_{i=1}^{I} \Big\{ d_{t_i,0}^{sell} \ln \Big(G(\theta_{0,1}^{sell}(z_{t_i}) - y_{t_i}|x_{t_i}) \lambda(t_i; x_{t_i}) \Big) \\
&\quad + \left(\sum_{s=1}^{S(z_{t_i})} d_{t_i,s}^{sell} \right) \ln \Big(\Big[G(\theta_{marginal}^{sell}(z_{t_i}) - y_{t_i}|x_{t_i}) - G(\theta_{0,1}^{sell}(z_{t_i}) - y_{t_i}|x_{t_i}) \Big] \lambda(t_i; x_{t_i}) \Big) \\
&\quad + d_{t_i,0}^{buy} \ln \Big(\Big[1 - G(\theta_{0,1}^{buy}(z_{t_i}) - y_{t_i}|x_{t_i}) \Big] \lambda(t_i; x_{t_i}) \Big) \\
&\quad + \left(\sum_{b=1}^{B(z_{t_i})} d_{t_i,b}^{buy} \right) \ln \Big(\Big[G(\theta_{0,1}^{buy}(z_{t_i}) - y_{t_i}|x_{t_i}) - G(\theta_{marginal}^{buy}(z_{t_i}) - y_{t_i}|x_{t_i}) \Big] \lambda(t_i; x_{t_i}) \Big) \\
&\quad - \int_{t_{i-1}}^{t_i} \Big[1 - G(\theta_{marginal}^{buy}(z_t) - y_t|x_t) + G(\theta_{marginal}^{sell}(z_t) - y_t|x_t) \Big] \lambda(t; x_t) dt \Big\}.
\end{aligned} \tag{77}$$

Assuming that the common value and the state vector only change in the case of order submission, the log-likelihood function simplifies to

$$
\begin{aligned}
& logL(.|z_{t_i}) \\
&= \sum_{i=1}^{I} \Big\{ d_{t_i,0}^{sell} \ln \Big(G(\theta_{0,1}^{sell}(z_{t_i}) - y_{t_i}|x_{t_i})\lambda(t_i;x_{t_i}) \Big) \\
&\quad + \left(\sum_{s=1}^{S(z_{t_i})} d_{t_i,s}^{sell} \right) \ln \Big(\Big[G(\theta_{marginal}^{sell}(z_{t_i}) - y_{t_i}|x_{t_i}) - G(\theta_{0,1}^{sell}(z_{t_i}) - y_{t_i}|x_{t_i}) \Big] \lambda(t_i;x_{t_i}) \Big) \\
&\quad + d_{t_i,0}^{buy} \ln \Big(\Big[1 - G(\theta_{0,1}^{buy}(z_{t_i}) - y_{t_i}|x_{t_i}) \Big] \lambda(t_i;x_{t_i}) \Big) \\
&\quad + \left(\sum_{b=1}^{B(z_{t_i})} d_{t_i,b}^{buy} \right) \ln \Big(\Big[G(\theta_{0,1}^{buy}(z_{t_i}) - y_{t_i}|x_{t_i}) - G(\theta_{marginal}^{buy}(z_{t_i}) - y_{t_i}|x_{t_i}) \Big] \lambda(t_i;x_{t_i}) \Big) \\
&\quad - \Big[1 - G(\theta_{marginal}^{buy}(z_{t_i}) - y_{t_i}|x_{t_i}) + G(\theta_{marginal}^{sell}(z_{t_i}) - y_{t_i}|x_{t_i}) \Big] \int_{t_{i-1}}^{t_i} \lambda(t;x_{t_i})dt \Big\} . \quad (78)
\end{aligned}
$$

Substituting the threshold valuations evaluated at the first step parameter estimates, the parameterization of the conditional private value distribution as well as that of the trader arrival rate into equation (78) allows to estimate $logL(.|z_t)$ by maximum likelihood. The remaining parameters to be estimated are the order submission cost c_0, the order execution cost c_e, the parameters that characterize the conditional private value distribution σ_1, σ_2, Γ and ρ as well as the Weibull parameters of the trader arrival rate model δ and η.[21] Given these estimates, the estimates of the threshold valuations $\hat{\theta}_{0,1}^{sell}(z_t)$, $\hat{\theta}_{0,1}^{buy}(z_t)$, $\hat{\theta}_{marginal}^{sell}(z_t)$ and $\hat{\theta}_{marginal}^{buy}(z_t)$ at every order submission can be derived that are needed for the computation of the final market efficiency measures.

Data Requirements

The structured recipe of the two step estimation procedure reveals which data requirements need to be fulfilled to actually conduct the estimation and to provide the estimates necessary for the subsequent computation of the market efficiency measures. These requirements are summarized in the following.

[21] The order execution and submission costs are estimated as a percentage of the common value in HMSS, i.e. c_e and c_0 are replaced by $c_e^p \cdot y_t$ and $c_0^p \cdot y_t$ in the log-likelihood function. Given the estimates $\hat{c}_e^p$ and $\hat{c}_0^p$, $\hat{c}_e$ and $\hat{c}_0$ are delivered by $\hat{c}_e^p \cdot y_t$ and $\hat{c}_0^p \cdot y_t$.

In order to estimate the ingredients of the final market efficiency measures for the Xetra LOB market, the availability of an event data set of a stock traded on Xetra with $i = 1, \ldots, I$ observations is necessary containing the time of the initial order submission t_i of each order, the failure time $t_i + \tau_i$ of each order, an indicator function for the cancelation of the ith order $I_{t_i}(\tau_{execute} > \tau_{cancel})$, an indicator function for the execution of the ith order $I_{t_i}(\tau_{execute} \leq \tau_{cancel})$, indicator functions for the type of the ith order $d^{sell}_{t_i,s}$ or $d^{buy}_{t_i,b}$ with $s = 0, \ldots, S(z_{t_i})$ and $b = 0, \ldots, B(z_{t_i})$, a proxy for the stock's common value at the time of the submission of the ith order y_{t_i} as well as at the failure time of the ith order $y_{t_i+\tau_i}$ and the corresponding exogenous as well as endogenous state variables $z_{t_i} = (x_{t_i}, \omega_{t_i})$.[22]

In the theoretical model the exogenous state variables x_{t_i} predict the trader arrival rates, the distribution of innovations to the common value and the conditional distribution of the traders' private values. In the empirical application they should be selected in a way that they are likely to be correlated with the traders' desire to change their portfolios and with innovations in the stocks' common value. In the ideal case the endogenous state variables w_{t_i} would include the entire LOB and any other variables known at t_i that help to predict the outcomes of order submissions at t_i. However, if the sample is relatively small it is better to use a smaller number of variables in the endogenous state vector w_{t_i}. Which specific variables are actually used as state variables for the Xetra application in the end depends on the choice of the stock for which the analysis is conducted.[23]

4.2 Computation of Market Efficiency Measures

The theoretical market efficiency measures from equations (26) and (27), once implemented empirically, allow to compute standardized and hence comparable measures that assess the efficiency of a real world LOB market. With the ingredients derived

[22] HMSS use as proxy for y_{t_i} a centered moving average of the midquotes $mq_{t_i} = (p^{buy}_{t_i,0} + p^{sell}_{t_i,0})/2$ over a 20-minute window. An adequate specification of y_t for the Xetra LOB data needs to be found.

[23] HMSS's use the TSX mining volatility as exogenous state variable for their application conducted for stocks that obey strong relations to the mining industry.

above, this paragraph deals with the computation of closed form solutions of these measures to actually document the efficiency of a real world LOB market like Xetra with numbers.

The estimate of the current gains from trade $\hat{CG}(z_t)$ is obtained by substituting the optimal order submission strategies of the sell side from equations (22) to (25) and the respective optimal order submission strategies of the buy side, both evaluated at the parameter estimates delivered in the preceding subparagraph, into the formula for the current gains from trade delivered in equation (29).[24] The resulting estimate equals

$$\begin{aligned}
&\hat{CG}(z_t)\\
&= \mathbb{E}\left[I(-\infty < y_t + u_t \leq \hat{\theta}^{sell}_{0,1}(z_t))(-u_t - \hat{c}_e - \hat{c}_0)|z_t\right]\\
&\quad + \sum_{s=1}^{S(z_t)-1} \mathbb{E}\left[I\left(\hat{\theta}^{sell}_{s-1,s}(z_t) \leq y_t + u_t \leq \hat{\theta}^{sell}_{s,s+1}(z_t)\right)\left(\hat{\psi}^{sell}_s(z_t)(-u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right]\\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{sell}_{S(z_t)-1,S(z_t)}(z_t) \leq y_t + u_t \leq \hat{\theta}^{sell}_{marginal}(z_t)\right)\left(\hat{\psi}^{sell}_{S(z_t)}(z_t)(-u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right]\\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) \leq y_t + u_t < \infty\right)(u_t - \hat{c}_e - \hat{c}_0)|z_t\right]\\
&\quad + \sum_{b=1}^{B(z_t)-1} \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{b,b+1}(z_t) \leq y_t + u_t \leq \hat{\theta}^{buy}_{b-1,b}(z_t)\right)\left(\hat{\psi}^{buy}_b(z_t)(u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right]\\
&\quad + \mathbb{E}\Big[I\left(\hat{\theta}^{buy}_{marginal}(z_t) \leq y_t + u_t \leq \hat{\theta}^{buy}_{B(z_t)-1,B(z_t)}(z_t)\right)\\
&\qquad \left(\hat{\psi}^{buy}_{B(z_t)}(z_t)(u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\Big]. \qquad (79)
\end{aligned}$$

[24] This estimate as well as the followinig estimates of the current gains from trade are computed at every order submission in the sample indexed by $i = 1, \ldots, I$. Nonetheless the index i is dropped for reasons of convenience.

Using solely the estimates of the execution probabilities for the marginal limit orders $\hat{\psi}^{buy}_{B(z_t)}(z_t)$ and $\hat{\psi}^{sell}_{S(z_t)}(z_t)$ for the execution probabilities of all limit orders $\psi^{buy}_b(z_t)$ and $\psi^{sell}_s(z_t)$ provides a lower bound for the estimate of the current gains from trade:

$$
\begin{aligned}
&\hat{CG}(z_t) \geq \hat{CG}_{lb}(z_t) \\
&= \mathbb{E}\left[I(-\infty < y_t + u_t \leq \hat{\theta}^{sell}_{0,1}(z_t))(-u_t - \hat{c}_e - \hat{c}_0)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) \leq y_t + u_t \leq \hat{\theta}^{sell}_{marginal}(z_t)\right)\left(\hat{\psi}^{sell}_{S(z_t)}(z_t)(-u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) \leq y_t + u_t < \infty\right)(u_t - \hat{c}_e - \hat{c}_0)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{marginal}(z_t) \leq y_t + u_t \leq \hat{\theta}^{buy}_{0,1}(z_t)\right)\left(\hat{\psi}^{buy}_{B(z_t)}(z_t)(u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right], \quad (80)
\end{aligned}
$$

since the true execution probabilities of orders submitted less far away from the best quotes than marginal limit orders are greater than the marginal execution probabilities computed for these orders.

Similarly, using solely the execution probabilities for the one-tick limit orders $\hat{\psi}^{buy}_1(z_t)$ and $\hat{\psi}^{sell}_1(z_t)$ for the execution probabilities of all limit orders $\psi^{buy}_b(z_t)$ and $\psi^{sell}_s(z_t)$ provides an upper bound for the estimate of the current gains from trade:

$$
\begin{aligned}
&\hat{CG}(z_t) \leq \hat{CG}_{ub}(z_t) \\
&= \mathbb{E}\left[I(-\infty < y_t + u_t \leq \hat{\theta}^{sell}_{0,1}(z_t))(-u_t - \hat{c}_e - \hat{c}_0)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) \leq y_t + u_t \leq \hat{\theta}^{sell}_{marginal}(z_t)\right)\left(\hat{\psi}^{sell}_1(z_t)(-u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) \leq y_t + u_t < \infty\right)(u_t - \hat{c}_e - \hat{c}_0)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{marginal}(z_t) \leq y_t + u_t \leq \hat{\theta}^{buy}_{0,1}(z_t)\right)\left(\hat{\psi}^{buy}_1(z_t)(u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right], \quad (81)
\end{aligned}
$$

since the true execution probabilities of orders submitted farther away from the best quotes than one-tick limit orders are smaller than one-tick execution probabilities computed for these orders.

Using the average execution probabilities, i.e. $\hat{\psi}^{buy}_{avg}(z_t) = 0.5 \cdot \left(\hat{\psi}^{buy}_1(z_t) + \hat{\psi}^{buy}_{B(z_t)}(z_t)\right)$ and $\hat{\psi}^{sell}_{avg}(z_t) = 0.5 \cdot \left(\hat{\psi}^{sell}_1(z_t) + \hat{\psi}^{sell}_{S(z_t)}(z_t)\right)$ for the execution probabilities of all limit orders $\psi^{buy}_b(z_t)$ and $\psi^{sell}_s(z_t)$ delivers an estimate of the average current gains from trade:

$$
\begin{aligned}
&\hat{CG}_{avg}(z_t) \\
&= \mathbb{E}\left[I(-\infty < y_t + u_t \leq \hat{\theta}^{sell}_{0,1}(z_t))(-u_t - \hat{c}_e - \hat{c}_0)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) \leq y_t + u_t \leq \hat{\theta}^{sell}_{marginal}(z_t)\right)\left(\hat{\psi}^{sell}_{avg}(z_t)(-u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) \leq y_t + u_t < \infty\right)(u_t - \hat{c}_e - \hat{c}_0)|z_t\right] \\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{marginal}(z_t) \leq y_t + u_t \leq \hat{\theta}^{buy}_{0,1}(z_t)\right)\left(\hat{\psi}^{buy}_{avg}(z_t)(u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right]. \qquad (82)
\end{aligned}
$$

To actually compute the estimates of the lower and upper bounds as well as the average current gains from trade conditional on z_t, equations (80) through (82) need to be expressed in terms of the private value distribution from equation (69). The closed form formulas of (80) through (82) as well as their derivation are provided in appendix A.2.1.[25]

The estimates of the lower bound $\hat{CG}_{lb}(z_t)$, upper bound $\hat{CG}_{ub}(z_t)$ and the average current gains from trade $\hat{CG}_{avg}(z_t)$ from equation (121) in appendix A.2.1 quantify how much a single trader arriving in t with unknown valuation for an asset is expected to contribute to the gains from trade in the respective market form in state z_t. To derive unconditional estimates, i.e. estimates that do not depend on z_t, expectations across all states z_t are taken. Taking expectations across z_t for the exemplary estimate of the lower bound of the current gains from trade delivers its unconditional expectation

$$
\mathbb{E}\left[\hat{CG}_{lb}(z_t)|\text{Trader arrives and submits an order}\right], \qquad (83)
$$

that is conditional on the arrival of a trader who submits an order, since samples delivered by electronic limit order book markets like Xetra do not capture the arrival of traders who abstain from submitting orders. With I being the total number of observa-

[25]The ready-to-use formulas illustrated in equation (121) are quite space-wasting and their appearance is superfluous at this passage. Nevertheless they are important for the planned empirical implementation.

tions in the sample, equation (83) can be estimated by its sample analogue

$$\hat{CG}_{lb} = \frac{1}{I}\sum_{i=1}^{I}\hat{CG}_{lb}(z_{t_i}). \tag{84}$$

The unconditional expected estimates of the upper bound $\hat{CG}_{ub}$ and the average of the current gains from trade $\hat{CG}_{avg}$ are computed similarly.[26]

The estimate of the maximum gains from trade can be derived by substituting the parameterization of the private value distribution from equation (69) evaluated at the parameter estimates into the formula for the maximum gains from trade in equation (37). The derivation of the resulting estimate

$$\begin{aligned}
&\hat{MaG}(x_t)\\
&= \hat{\rho}\left(2\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right) - (\hat{c}_e+\hat{c}_0)2\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right)\\
&\quad +(1-\hat{\rho})\left(2\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right) - (\hat{c}_e+\hat{c}_0)2\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right).
\end{aligned} \tag{85}$$

can be retraced in appendix A.2.2.[27] Its unconditional counterpart can be derived as in the case of the current gains from trade and equals

$$\hat{MaG} = \frac{1}{I}\sum_{i=1}^{I}\hat{MaG}(x_{t_i}), \tag{86}$$

with I being the total number of observations in the sample.

The estimate of the monopoly gains from trade can be derived by substituting the parameterization of the private value distribution from equation (69) evaluated at the parameter estimates and the corresponding density function first into the monopolist's optimal quotes and second via the traders' optimal order submission strategies from equations (44) and (45) into the formula of the monopoly gains from trade given in

[26]Due to the fact that the sample does not capture trader arrivals that are associated with traders who abstain from trading, equation (83) and the respective measures for the upper bound and the average of the current gains from trade as well as the maximum and the monopoly gains from trade are downward biased. To correct for this bias, HMSS develop a method to reweight their sample. For shortage of space, this thesis abstains from delivering the reweighting procedure as it is explicitly described in HMSS.

[27]Note that equation (85) does not report $\hat{MaG}(z_t)$ as a percentage of the common value as in HMSS but it delivers $\hat{MaG}(z_t)$ in absolute terms.

equation (46). The derivation of the resulting estimate

$$
\begin{aligned}
&\hat{MoG}(x_t) \\
&= \hat{\rho}\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right) \\
&\quad - \left(\hat{\rho}\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right)(\hat{c}_e + \hat{c}_0) \\
&\quad + \hat{\rho}\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right) \\
&\quad - \left(\hat{\rho}\left(1 - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) + (1-\hat{\rho})\left(1 - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right)\right) \\
&\quad (\hat{c}_e + \hat{c}_0) \qquad (87)
\end{aligned}
$$

can be followed in appendix A.2.3. Its unconditional analogue is obtained by the sample average

$$\hat{MoG} = \frac{1}{I}\sum_{i=1}^{I} \hat{MoG}(x_{t_i}), \qquad (88)$$

with I being the total number of observations in the sample.

With numbers assessed to the unconditional expected estimates of the upper and lower bounds as well as to the average of the current gains from trade and with the unconditional expected estimates of the maximum as well as of the monopoly gains from trade derived, the upper and lower bounds as well as the average of the final market efficiency measures can be computed as follows:

$$\widehat{\text{Market efficiency measure I}}_{lb} = \frac{\hat{CG}_{lb}}{\hat{MaG}} \times 100[\%], \qquad (89)$$

$$\widehat{\text{Market efficiency measure I}}_{ub} = \frac{\hat{CG}_{ub}}{\hat{MaG}} \times 100[\%], \qquad (90)$$

$$\widehat{\text{Market efficiency measure I}}_{avg} = \frac{\hat{CG}_{avg}}{\hat{MaG}} \times 100[\%], \qquad (91)$$

$$\widehat{\text{Market efficiency measure II}}_{lb} = \frac{\hat{CG}_{lb}}{\hat{MoG}} \times 100[\%], \qquad (92)$$

$$\widehat{\text{Market efficiency measure II}}_{ub} = \frac{\hat{CG}_{ub}}{\hat{MoG}} \times 100[\%], \qquad (93)$$

$$\widehat{\text{Market efficiency measure II}}_{avg} = \frac{\hat{CG}_{avg}}{\hat{MoG}} \times 100[\%]. \qquad (94)$$

5 Empirical Caveats and Extensions

The quality of the market efficiency measures derived and implemented above depends on how good the data matches the theoretical assumptions of the model. Which assumptions are particularly important and how one can check whether the data admits to use these assumptions as reasonable approximations is discussed in the following.

First, the theoretical model is based on the assumption that no hidden limit order (also called iceberg orders) enter the LOB although many LOB markets like Xetra for example allow limit order traders to submit such orders. Whether the assumption of no hidden limit orders can be maintained or not depends on how frequently traders make use of them in the market under investigation. To check for the importance of hidden limit orders in the sample under investigation, compute summary statistics of order submissions that reveal the percentage of hidden limit orders. If the fraction of hidden limit orders in the sample is sufficiently small, the assumption of no hidden limit orders can be maintained. Otherwise HMSS's base model needs to modified in order to take hidden limit orders into account for the derivation of the market efficiency measures.

Second, in the theoretical model order quantity is normalized to unity which is reasonable under the assumption that orders fully quit the LOB either as consequence of execution or due to cancelation. On the contrary, in real world LOB markets like Xetra partial executions occur. HMSS propose to deal with partial executions by attaching a partially executed order to the group of fully executed orders if at least 50% of its quantity is executed and to the group of fully canceled orders otherwise. Again descriptive statistics help to assess whether this procedure can be maintained as a reasonable approximation of the data from the Xetra trading system: the greater the average percentage of the submitted order quantity within the so-arranged group of fully executed orders and the smaller the average percentage of the submitted order quantity within the group of fully canceled orders, the better this approximation holds.

Third, the maximum likelihood estimations of the competing risks models for the latent cancelation and execution times require sufficiently large samples in order to ensure consistent results. The sample size may in particular be a problem for the subsets of marginal buy and sell limit orders which presumably will occur quite infrequently. If the subsamples of marginal buy or sell limit orders in the Xetra data set happen to be too small to consistently estimate the execution and cancelation hazard rates, eventually marginal limit orders need to be combined with orders up to a certain number of ticks away from the marginal prices.

Fourth, although orders are allowed to last for multiple periods in the theoretical model, the computation of the estimates of the execution probabilities compare equation (59) requires the introduction of a censoring time span ΔT. The censoring implies that orders lasting longer than ΔT in the LOB are treated as canceled orders. To identify ΔT for the sample under investigation, descriptive statistics revealing the distribution of the time to execution of orders are useful. HMSS for example set ΔT to $48,600$ seconds which corresponds to two trading days. They argue that this is a reasonable assumption since less than 1% of order executions occur later than two trading days after the time of the order submission in their sample.

Taking these empirical caveats into account should result in market efficiency measures that reflect the quality of a real world LOB market like Xetra quite well.

6 Conclusion

Motivated by the prominent role of electronic LOB markets in today's stock market environment, this paper provides the basis for understanding, reconstructing and adopting HMSS's methodology for estimating the gains from trade to the Xetra LOB market at the FSE in order to evaluate its performance in this respect. Therefore this paper looked deeply into HMSS's base model and provided a structured recipe for the planned implementation with Xetra LOB data. The contribution of this paper lies in the modification of HMSS's methodology with respect to the particularities of the Xetra trading system that are not yet considered in HMSS's base model. The necessary modifications, as expressed in terms of empirical caveats, are substantial to derive unbiased market efficiency measures for Xetra in the end.

Future work should first and foremost focus on the realization of the project to estimate the gains from trade for the Xetra LOB market alongside the structured recipe provided above. However, HMSS's original estimation procedure, which remains untouched in this work, and hence also the structured recipe provided above leave a great margin for improvements concerning the choice of econometric tools. As a consequence, future work should additionally focus on developing an advanced econometric foundation that will reflect real world LOB market characteristics even better. A third interesting direction of future research is to advance the methodology for estimating the gains from trade to its application not only to LOB markets but to other important market designs like the hybrid trading system implemented at the New York Stock Exchange. Such an extension would enable researches to directly compare the most important trading designs in terms of the gains from trade and to answer the question of whether particular market designs perform better than others in this respect.

A Appendix

A.1 Competing Risks Model Essentials

Competing risks models provide a method for data analysis when there is a single, possibly censored failure time FT on each of $i = 1, \ldots, I$ study objects that occurs as consequence of $e = 1, \ldots, E$ distinct causes. The failure type and the failure time are observed fulfilling $FT = \min(FT_1, \ldots, FT_E)$.

A.1.1 Derivation of conditional log-likelihood function of competing risks model

Suppose I study objects indexed by $i = 1, \ldots, I$ give rise to data $(t_i, \delta_{i,e}, Q_i^{cr})$ for $e = 1, \ldots, E$ possible failure types where t_i is the time of the ith failure, $\delta_{i,e}$ is an indicator function for the ith failure type, and Q_i^{cr} is the associated conditioning information. On the basis of the competing risks model theory provided in Kalbfleisch and Prentice (2002), chapter 8, and Lancaster (1990), chapter 5, HMSS show that the conditional likelihood function of an independent competing risks model with $e = 1, \ldots, E$ distinct failure types and independent latent failure times FT_e looks as follows

$$L(.|Q^{cr}) = \prod_{i=1}^{I} \left\{ \prod_{e=1}^{E} h_e(t_i; Q_i^{cr})^{\delta_{i,e}} \exp\left(-\int_0^{t_i} \sum_{e=1}^{E} h_e(s; Q_i^{cr}) ds \right) \right\}. \tag{95}$$

Taking logarithms delivers the log-likelihood function

$$logL(.|Q^{cr}) = \sum_{i=1}^{I} \left\{ \sum_{e=1}^{E} \delta_{i,e} \ln\left(h_e(t_i; Q_i^{cr})\right) - \int_0^{t_i} \sum_{e=1}^{E} h_e(s; Q_i^{cr}) ds \right\}, \tag{96}$$

where $h_e(t; Q^{cr})$ is the hazard rate for the eth latent failure time FT_e

$$\begin{aligned} h_e(t; Q^{cr}) &= \lim_{\Delta t \to 0} \frac{\Pr(FT_e \in [t, t+\Delta t) | FT \geq t, Q^{cr})}{\Delta t} && (97) \\ &= \frac{f_e(t|Q^{cr})}{1 - F_e(t|Q^{cr})} && (98) \end{aligned}$$

with $F_e(t|Q^{cr})$ denoting the distribution of the eth latent failure time and $f_e(t|Q^{cr})$ the associated density. These results are used to compute the log-likelihood functions in the first and the second step estimation in section 4.1.

A.2 Computation of Market Efficiency Measures

A.2.1 Current Gains from Trade

The final formulas of the estimates of the lower and upper bounds as well as the average current gains from trade are derived by substituting the parameterization of the private value distribution from equation (69) evaluated at the parameter estimates

$$G(u|x_t) = \hat{\rho}\Phi\left(\frac{u}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{u}{\hat{\sigma}_2^*(y_t, x_t)}\right) \tag{99}$$

with

$$\begin{aligned} \hat{\sigma}_1^*(y_t, x_t) &= y_t \hat{\sigma}_1 \exp(x_t' \hat{\Gamma}), && (100) \\ \hat{\sigma}_2^*(y_t, x_t) &= y_t \hat{\sigma}_2 \exp(x_t' \hat{\Gamma}), && (101) \end{aligned}$$

into equations (80) through (82). Consider as illustrative example the estimate of the lower bound of the current gains from trade:

$$\begin{aligned}
&\hat{CG}_{lb}(z_t)\\
&= \mathbb{E}\left[I(-\infty < y_t + u_t \le \hat{\theta}^{sell}_{0,1}(z_t))(-u_t - \hat{c}_e - \hat{c}_0)|z_t\right]\\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) \le y_t + u_t \le \hat{\theta}^{sell}_{marginal}(z_t)\right)\left(\hat{\psi}^{sell}_{S(z_t)}(z_t)(-u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right]\\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) \le y_t + u_t < \infty\right)(u_t - \hat{c}_e - \hat{c}_0)|z_t\right]\\
&\quad + \mathbb{E}\left[I\left(\hat{\theta}^{buy}_{marginal}(z_t) \le y_t + u_t \le \hat{\theta}^{buy}_{0,1}(z_t)\right)\left(\hat{\psi}^{buy}_{B(z_t)}(z_t)(u_t - \hat{c}_e) - \hat{c}_0\right)|z_t\right], \qquad (102)
\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}
&\hat{CG}_{lb}(z_t)\\
&= \left|-\mathbb{E}\left[I(-\infty - y_t < u_t \le \hat{\theta}^{sell}_{0,1}(z_t) - y_t)u_t|z_t\right]\right| && (103)\\
&\quad -\mathbb{E}\left[I(-\infty - y_t < u_t \le \hat{\theta}^{sell}_{0,1}(z_t) - y_t)|z_t\right](\hat{c}_e + \hat{c}_0) && (104)\\
&\quad + \left|-\mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) - y_t \le u_t \le \hat{\theta}^{sell}_{marginal}(z_t) - y_t\right)u_t|z_t\right]\hat{\psi}^{sell}_{S(z_t)}(z_t)\right| && (105)\\
&\quad -\mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) - y_t \le u_t \le \hat{\theta}^{sell}_{marginal}(z_t) - y_t\right)|z_t\right]\left(\hat{\psi}^{sell}_{S(z_t)}(z_t)\hat{c}_e + \hat{c}_0\right) && (106)\\
&\quad + \left|\mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) - y_t \le u_t < \infty - y_t\right)u_t|z_t\right]\right| && (107)\\
&\quad -\mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) - y_t \le u_t < \infty - y_t\right)|z_t\right](\hat{c}_e + \hat{c}_0) && (108)\\
&\quad + \left|\mathbb{E}\left[I\left(\hat{\theta}^{buy}_{marginal}(z_t) - y_t \le u_t \le \hat{\theta}^{buy}_{0,1}(z_t) - y_t\right)u_t|z_t\right]\hat{\psi}^{buy}_{B(z_t)}(z_t)\right| && (109)\\
&\quad -\mathbb{E}\left[I\left(\hat{\theta}^{buy}_{marginal}(z_t) - y_t \le u_t \le \hat{\theta}^{buy}_{0,1}(z_t) - y_t\right)|z_t\right]\left(\hat{\psi}^{buy}_{B(z_t)}(z_t)\hat{c}_e + \hat{c}_0\right). && (110)
\end{aligned}$$

The following transformation rules for the random variable u being distributed as a mixture of two normal distributions with mean zero and standard deviations $\hat{\sigma}^*_1(y_t, x_t)$, $\hat{\sigma}^*_2(y_t, x_t)$, weighting factor $\hat{\rho}$, and $I(a \le u \le b)$ an indicator function that takes the value

1 for $a \le u \le b$ hold:

$$\begin{aligned}
&\mathbb{E}\left[I(a \le u \le b)u\right] \\
&= \hat{\rho}\left(\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{b}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{a}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{b}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{a}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right),
\end{aligned} \tag{111}$$

$$\begin{aligned}
&\mathbb{E}\left[I(a \le u \le b)\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{b}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \Phi\left(\frac{a}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{b}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \Phi\left(\frac{a}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right).
\end{aligned} \tag{112}$$

Applying (111) to (103) delivers

$$\begin{aligned}
&\mathbb{E}\left[I(-\infty - y_t < u_t \le \hat{\theta}_{0,1}^{sell}(z_t) - y_t)u_t|z_t\right] \\
&= \hat{\rho}\left(\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{-\infty - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{-\infty - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right) \\
&= \hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right).
\end{aligned} \tag{113}$$

Applying (112) to (104) delivers

$$\begin{aligned}
&\mathbb{E}\left[I(-\infty - y_t < u_t \le \hat{\theta}_{0,1}^{sell}(z_t) - y_t)|z_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \Phi\left(\frac{-\infty - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \Phi\left(\frac{-\infty - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right) \\
&= \hat{\rho}\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right).
\end{aligned} \tag{114}$$

Applying (111) to (105) delivers

$$\begin{aligned}
&\mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) - y_t \le u_t \le \hat{\theta}^{sell}_{marginal}(z_t) - y_t\right) u_t | z_t\right] \\
&= \hat{\rho}\left(\hat{\sigma}^*_1(y_t, x_t)\phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t) - y_t}{\hat{\sigma}^*_1(y_t, x_t)}\right) - \hat{\sigma}^*_1(y_t, x_t)\phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_1(y_t, x_t)}\right)\right) \\
&\quad + (1-\hat{\rho})\left(\hat{\sigma}^*_2(y_t, x_t)\phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t) - y_t}{\hat{\sigma}^*_2(y_t, x_t)}\right) - \hat{\sigma}^*_2(y_t, x_t)\phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_2(y_t, x_t)}\right)\right) \qquad (115)
\end{aligned}$$

Applying (112) to (106) delivers

$$\begin{aligned}
&\mathbb{E}\left[I\left(\hat{\theta}^{sell}_{0,1}(z_t) - y_t \le u_t \le \hat{\theta}^{sell}_{marginal}(z_t) - y_t\right) | z_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t) - y_t}{\hat{\sigma}^*_1(y_t, x_t)}\right) - \Phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_1(y_t, x_t)}\right)\right) \\
&\quad + (1-\hat{\rho})\left(\Phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t) - y_t}{\hat{\sigma}^*_2(y_t, x_t)}\right) - \Phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_2(y_t, x_t)}\right)\right). \qquad (116)
\end{aligned}$$

Applying (111) to (107) delivers

$$\begin{aligned}
&\mathbb{E}\left[I\left(\hat{\theta}^{buy}_{0,1}(z_t) - y_t \le u_t < \infty - y_t\right) u_t | z_t\right] \\
&= \hat{\rho}\left(\hat{\sigma}^*_1(y_t, x_t)\phi\left(\frac{\infty - y_t}{\hat{\sigma}^*_1(y_t, x_t)}\right) - \hat{\sigma}^*_1(y_t, x_t)\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_1(y_t, x_t)}\right)\right) \\
&\quad + (1-\hat{\rho})\left(\hat{\sigma}^*_2(y_t, x_t)\phi\left(\frac{\infty - y_t}{\hat{\sigma}^*_2(y_t, x_t)}\right) - \hat{\sigma}^*_2(y_t, x_t)\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_2(y_t, x_t)}\right)\right) \\
&= -\left(\hat{\rho}\hat{\sigma}^*_1(y_t, x_t)\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_1(y_t, x_t)}\right) + (1-\hat{\rho})\hat{\sigma}^*_2(y_t, x_t)\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t) - y_t}{\hat{\sigma}^*_2(y_t, x_t)}\right)\right) \qquad (117)
\end{aligned}$$

Applying (112) to (108) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I\left(\hat{\theta}_{0,1}^{buy}(z_t) - y_t \leq u_t < \infty - y_t\right)|z_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{\infty - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{\infty - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right) \\
&= \hat{\rho}\left(1 - \Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\left(1 - \Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right). \qquad (118)
\end{aligned}
$$

Applying (111) to (109) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I\left(\hat{\theta}_{marginal}^{buy}(z_t) - y_t \leq u_t \leq \hat{\theta}_{0,1}^{buy}(z_t) - y_t\right)u_t|z_t\right] \\
&= \hat{\rho}\left(\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) \\
&\quad -(1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right). \qquad (119)
\end{aligned}
$$

Applying (112) to (110) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I\left(\hat{\theta}_{marginal}^{buy}(z_t) - y_t \leq u_t \leq \hat{\theta}_{0,1}^{buy}(z_t) - y_t\right)|z_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t) - y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t) - y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right). \qquad (120)
\end{aligned}
$$

Resubstituting (113) through (120) into (103) through (110) delivers the closed form solution of the estimate of the lower bound of the current gains from trade:

$$
\begin{aligned}
&\hat{CG}_{lb}(z_t) \\
&= \hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) \\
&\quad - \left(\hat{\rho}\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)(\hat{c}_e+\hat{c}_0) \\
&\quad + \left\{\hat{\rho}\left(\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\right. \\
&\quad \left.\left(\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right\}\hat{\psi}_{S(z_t)}^{sell}(z_t) \\
&\quad - \left\{\hat{\rho}\left(\Phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\right. \\
&\quad \left.\left(\Phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right\}\left(\hat{\psi}_{S(z_t)}^{sell}(z_t)\hat{c}_e+\hat{c}_0\right) \\
&\quad + \left(\hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right) \\
&\quad - \left(\hat{\rho}\left(1-\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\left(1-\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right)(\hat{c}_e+\hat{c}_0) \\
&\quad + \left\{\hat{\rho}\left(\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\right. \\
&\quad \left.\left(\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right\}\hat{\psi}_{B(z_t)}^{buy}(z_t) \\
&\quad - \left\{\hat{\rho}\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\right. \\
&\quad \left.\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) - \Phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right\}\left(\hat{\psi}_{B(z_t)}^{buy}(z_t)\hat{c}_e+\hat{c}_0\right) \qquad (121)
\end{aligned}
$$

The closed form solutions for the upper bound $\hat{CG}_{ub}(z_t)$ and the average current gains from trade $\hat{CG}_{avg}(z_t)$ are as given in equation (121) with the estimates of the execution probabilities $\hat{\psi}_{S(z_t)}^{buy}(z_t)$ and $\hat{\psi}_{S(z_t)}^{sell}(z_t)$ replaced by their appropriate counterparts

$\hat{\psi}_1^{buy}(z_t)$ and $\hat{\psi}_1^{sell}(z_t)$ or $\hat{\psi}_{avg}^{buy}(z_t)$ and $\hat{\psi}_{avg}^{sell}(z_t)$.

Substituting $y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})$ for $\hat{\sigma}_1^*(y_t,x_t)$ and $y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})$ for $\hat{\sigma}_2^*(y_t,x_t)$ as well as $\hat{c}_e^p\cdot y_t$ for $\hat{c}_e$ and $\hat{c}_0^p\cdot y_t$ for $\hat{c}_0$, equation (121) can be written as

$$
\begin{aligned}
&\hat{CG}_{lb}(z_t)\\
&=\hat{\rho}y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)+(1-\hat{\rho})y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\\
&\quad-\left(\hat{\rho}\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)+(1-\hat{\rho})\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)(\hat{c}_e^p+\hat{c}_0^p)y_t\\
&\quad+\left\{\hat{\rho}\left(y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)-y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right)+(1-\hat{\rho})\right.\\
&\quad\left.\left(y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)-y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\right\}\hat{\psi}_{S(z_t)}^{sell}(z_t)\\
&\quad-\left\{\hat{\rho}\left(\Phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)-\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right)+(1-\hat{\rho})\right.\\
&\quad\left.\left(\Phi\left(\frac{\hat{\theta}_{marginal}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)-\Phi\left(\frac{\hat{\theta}_{0,1}^{sell}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\right\}\left(\hat{\psi}_{S(z_t)}^{sell}(z_t)\hat{c}_e^p+\hat{c}_0^p\right)y_t\\
&\quad+\left(\hat{\rho}y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)+(1-\hat{\rho})y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\\
&\quad-\left(\hat{\rho}\left(1-\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right)+(1-\hat{\rho})\left(1-\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\right)(\hat{c}_e^p+\hat{c}_0^p)y_t\\
&\quad+\left\{\hat{\rho}\left(y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)-y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right)+(1-\hat{\rho})\right.\\
&\quad\left.\left(y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)-y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\right\}\hat{\psi}_{B(z_t)}^{buy}(z_t)\\
&\quad-\left\{\hat{\rho}\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)-\Phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right)+(1-\hat{\rho})\right.\\
&\quad\left.\left(\Phi\left(\frac{\hat{\theta}_{0,1}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)-\Phi\left(\frac{\hat{\theta}_{marginal}^{buy}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\right\}\left(\hat{\psi}_{B(z_t)}^{buy}(z_t)\hat{c}_e^p+\hat{c}_0^p\right)y_t \qquad (122)
\end{aligned}
$$

such that the estimate of the lower bound of the current gains from trade as a percentage of the common value equals

$$
\begin{aligned}
&\frac{\hat{CG}_{lb}(z_t)}{y_t} \\
&= \hat{\rho}\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) \\
&\quad - \left(\hat{\rho}\Phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)(\hat{c}^p_e+\hat{c}^p_0) \\
&\quad + \Bigg\{\hat{\rho}\left(\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) - \hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right) + (1-\hat{\rho}) \\
&\quad \left(\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) - \hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\Bigg\}\hat{\psi}^{sell}_{S(z_t)}(z_t) \\
&\quad - \Bigg\{\hat{\rho}\left(\Phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) - \Phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right) + (1-\hat{\rho}) \\
&\quad \left(\Phi\left(\frac{\hat{\theta}^{sell}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) - \Phi\left(\frac{\hat{\theta}^{sell}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\Bigg\}\left(\hat{\psi}^{sell}_{S(z_t)}(z_t)\hat{c}^p_e+\hat{c}^p_0\right) \\
&\quad + \left(\hat{\rho}\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right) \\
&\quad - \left(\hat{\rho}\left(1-\Phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right) + (1-\hat{\rho})\left(1-\Phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\right)(\hat{c}^p_e+\hat{c}^p_0) \\
&\quad + \Bigg\{\hat{\rho}\left(\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) - \hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{buy}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right) + (1-\hat{\rho}) \\
&\quad \left(\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) - \hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{\theta}^{buy}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\Bigg\}\hat{\psi}^{buy}_{B(z_t)}(z_t) \\
&\quad - \Bigg\{\hat{\rho}\left(\Phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) - \Phi\left(\frac{\hat{\theta}^{buy}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right) + (1-\hat{\rho}) \\
&\quad \left(\Phi\left(\frac{\hat{\theta}^{buy}_{0,1}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) - \Phi\left(\frac{\hat{\theta}^{buy}_{marginal}(z_t)-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\Bigg\}\left(\hat{\psi}^{buy}_{B(z_t)}(z_t)\hat{c}^p_e+\hat{c}^p_0\right).
\end{aligned}
\tag{123}
$$

Again, the closed form solutions for the upper bound $\hat{CG}_{ub}(z_t)$ and the average current gains from trade $\hat{CG}_{avg}(z_t)$ as a percentage of the common value are as given in equation (123) with the estimates of the execution probabilities $\hat{\psi}^{buy}_{S(z_t)}(z_t)$ and $\hat{\psi}^{sell}_{S(z_t)}(z_t)$ replaced by their appropriate counterparts $\hat{\psi}^{buy}_1(z_t)$ and $\hat{\psi}^{sell}_1(z_t)$ or $\hat{\psi}^{buy}_{avg}(z_t)$ and $\hat{\psi}^{sell}_{avg}(z_t)$.

A.2.2 Maximum Gains from Trade

The estimate of the maximum gains from trade can be derived by substituting the parameterization of the private value distribution evaluated at the parameter estimates into the formula for the maximum gains from trade

$$MaG(x_t) \quad = \quad \mathbb{E}\left[\begin{array}{c} I^{sell*}(u_t;x_t)(-u_t-c_e-c_0) \\ +I^{buy*}(u_t;x_t)(u_t-c_e-c_0) \end{array}\middle| x_t\right], \tag{124}$$

which, written more extensively, looks as follows:

$$\begin{aligned}
& MaG(x_t) \\
& = \mathbb{E}\left[I^{sell*}(u_t;x_t)(-u_t-c_e-c_0)+I^{buy*}(u_t;x_t)(u_t-c_e-c_0)|x_t\right] \\
& = \mathbb{E}\left[I^{sell*}(-\infty<u_t\le -c_e-c_0;x_t)(-u_t-c_e-c_0)|x_t\right] \\
& \quad +\mathbb{E}\left[I^{buy*}(c_e+c_0\le u_t<\infty;x_t)(u_t-c_e-c_0)|x_t\right] \\
& = \left|-\mathbb{E}\left[I^{sell*}(-\infty<u_t\le -c_e-c_0;x_t)u_t|x_t\right]\right| && (125) \\
& \quad -\mathbb{E}\left[I^{sell*}(-\infty<u_t\le -c_e-c_0;x_t)|x_t\right](c_e+c_0) && (126) \\
& \quad +\left|\mathbb{E}\left[I^{buy*}(c_e+c_0\le u_t<\infty;x_t)u_t|x_t\right]\right| && (127) \\
& \quad -\mathbb{E}\left[I^{buy*}(c_e+c_0\le u_t<\infty;x_t)|x_t\right](c_e+c_0). && (128)
\end{aligned}$$

Applying (111) to (125) delivers

$$\begin{aligned}
& \mathbb{E}\left[I^{sell*}(-\infty<u_t\le -\hat{c}_e-\hat{c}_0;x_t)u_t|x_t\right] \\
& = \hat{\rho}\left(\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\sigma_1^*(y_t,x_t)}\right)-\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{-\infty}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
& \quad +(1-\hat{\rho})\left(\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)-\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{-\infty}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right) \\
& = \hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)+(1-\hat{\rho})\sigma_2^*(y_t,x_t)\phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right). && (129)
\end{aligned}$$

Applying (112) to (126) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I^{sell*}(-\infty < u_t \leq -\hat{c}_e - \hat{c}_0; x_t)|x_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{-\hat{c}_e - \hat{c}_0}{\hat{\sigma}_1^*(y_t, x_t)}\right) - \Phi\left(\frac{-\infty}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{-\hat{c}_e - \hat{c}_0}{\hat{\sigma}_2^*(y_t, x_t)}\right) - \Phi\left(\frac{-\infty}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right) \\
&= \hat{\rho}\Phi\left(\frac{-\hat{c}_e - \hat{c}_0}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{-\hat{c}_e - \hat{c}_0}{\sigma_2^*(y_t, x_t)}\right).
\end{aligned}
\tag{130}
$$

Applying (111) to (127) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I^{buy*}(\hat{c}_e + \hat{c}_0 \leq u_t < \infty; x_t)u_t|x_t\right] \\
&= \hat{\rho}\left(\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\infty}{\hat{\sigma}_1^*(y_t, x_t)}\right) - \hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\infty}{\hat{\sigma}_2^*(y_t, x_t)}\right) - \hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right) \\
&= -\left(\hat{\rho}\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right).
\end{aligned}
\tag{131}
$$

Applying (112) to (128) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I^{buy*}(\hat{c}_e + \hat{c}_0 \leq u_t < \infty; x_t)|x_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{\infty}{\hat{\sigma}_1^*(y_t, x_t)}\right) - \Phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{\infty}{\hat{\sigma}_2^*(y_t, x_t)}\right) - \Phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right) \\
&= \hat{\rho}\left(1 - \Phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) + (1-\hat{\rho})\left(1 - \Phi\left(\frac{\hat{c}_e + \hat{c}_0}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right).
\end{aligned}
\tag{132}
$$

Substituting (129) to (132) into (125) to (128) and collecting terms delivers

$$
\begin{aligned}
&\hat{MaG}(x_t) \\
&= \left|-\hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)-(1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right| \\
&\quad+\left(-\hat{\rho}\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)-(1-\hat{\rho})\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)(\hat{c}_e+\hat{c}_0) \\
&\quad\left|-\hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)-(1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right| \\
&\quad+\left(-\hat{\rho}\left(1-\Phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right)-(1-\hat{\rho})\left(1-\Phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right)(\hat{c}_e+\hat{c}_0) \\
&= \hat{\rho}2\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)+(1-\hat{\rho})2\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right) \\
&\quad\left(-\hat{\rho}2\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)-(1-\hat{\rho})2\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)(\hat{c}_e+\hat{c}_0) \\
&= \hat{\rho}\left(2\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)-(\hat{c}_e+\hat{c}_0)2\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) \\
&\quad+(1-\hat{\rho})\left(2\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{c}_e+\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)-(\hat{c}_e+\hat{c}_0)2\Phi\left(\frac{-\hat{c}_e-\hat{c}_0}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right). \qquad (133)
\end{aligned}
$$

In the first step the terms from above are substituted and the curved brackets are multiplied out partially. In the second step the relationships $\phi(u) = \phi(-u)$ and $\Phi(-u) = 1 - \Phi(u)$ are applied to collect terms.

Substituting $y_t\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})$ for $\hat{\sigma}_1^*(y_t, x_t)$ and $y_t\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})$ for $\hat{\sigma}_2^*(y_t, x_t)$ as well as $\hat{c}_e^p \cdot y_t$ for $\hat{c}_e$ and $\hat{c}_0^p \cdot y_t$ for $\hat{c}_0$, equation (133) can be written as

$$
\begin{aligned}
&\hat{MaG}(x_t) \\
&= \hat{\rho}\left(2y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{c}_e^p+\hat{c}_0^p}{\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)-(\hat{c}_e^p+\hat{c}_0^p)y_t2\Phi\left(\frac{-\hat{c}_e^p-\hat{c}_0^p}{\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right) \\
&\quad+(1-\hat{\rho})2y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{c}_e^p+\hat{c}_0^p}{\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) \\
&\quad-(1-\hat{\rho})(\hat{c}_e^p+\hat{c}_0^p)2y_t\Phi\left(\frac{-\hat{c}_e^p-\hat{c}_0^p}{\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) \qquad (134)
\end{aligned}
$$

such that the estimate of the maximum gains as a percentage of the common value equals

$$\frac{\hat{MaG}(x_t)}{y_t}$$
$$= \hat{\rho}\left(2\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{c}_e^p + \hat{c}_0^p}{\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})}\right) - (\hat{c}_e^p + \hat{c}_0^p)2\Phi\left(\frac{-\hat{c}_e^p - \hat{c}_0^p}{\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})}\right)\right)$$
$$+(1-\hat{\rho})\left(2\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{c}_e^p + \hat{c}_0^p}{\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})}\right) - (\hat{c}_e^p + \hat{c}_0^p)2\Phi\left(\frac{-\hat{c}_e^p - \hat{c}_0^p}{\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})}\right)\right) \quad (135)$$

which corresponds to the closed form solution of the maximum gains from trade provided in the paper of HMSS.

A.2.3 Monopoly Gains from Trade

The estimate of the monopoly gains from trade is obtained by substituting the parameterization of the private value distribution evaluated at the parameter estimates from (99) and the corresponding density function first into the monopolist's optimal quotes and second via the traders' optimal order submission strategies from (44) and (45) into the formula of the monopoly gains from trade from (46).

The estimates of the monopolist's optimal quotes hence are obtained by solving

$$\begin{aligned}
\hat{b}_t^{m*} &= y_t - \frac{G(\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t|x_t)}{g(\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t|x_t)} \\
&= y_t - \frac{\hat{\rho}\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)}{\hat{\rho}\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)}, \quad (136)
\end{aligned}$$

$$\begin{aligned}
\hat{a}_t^{m*} &= y_t + \frac{1 - G(\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t|x_t)}{g(\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t|x_t)} \\
&= y_t + \frac{1 - \left(\hat{\rho}\Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right)}{\hat{\rho}\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)} \quad (137)
\end{aligned}$$

for $\hat{b}_t^{m*}$ and $\hat{a}_t^{m*}$.

The monopoly gains from trade

$$MoG(x_t) = \mathbb{E}\left[\begin{array}{c} I^{m,sell}(b_t^{m*}; u_t; x_t)(-u_t - c_e - c_0) \\ +I^{m,buy}(a_t^{m*}; u_t; x_t)(u_t - c_e - c_0) \end{array}\middle| x_t\right] \tag{138}$$

can be rewritten as

$$\begin{aligned}
& MoG(x_t) \\
&= \mathbb{E}\left[I^{m,sell}(-\infty < u_t \leq b_t^{m*} - c_e - c_0 - y_t; x_t)(-u_t - c_e - c_0)|x_t\right] \\
&\quad +\mathbb{E}\left[I^{m,buy}(a_t^{m*} + c_e + c_0 - y_t \leq u_t < \infty; x_t)(u_t - c_e - c_0)|x_t\right] \\
&= \left|-\mathbb{E}\left[I^{m,sell}(-\infty < u_t \leq b_t^{m*} - c_e - c_0 - y_t; x_t)u_t|x_t\right]\right| && (139) \\
&\quad -\mathbb{E}\left[I^{m,sell}(-\infty < u_t \leq b_t^{m*} - c_e - c_0 - y_t; x_t)|x_t\right](c_e + c_0) && (140) \\
&\quad +\left|\mathbb{E}\left[I^{m,buy}(a_t^{m*} + c_e + c_0 - y_t \leq u_t < \infty; x_t)u_t|x_t\right]\right| && (141) \\
&\quad -\mathbb{E}\left[I^{m,buy}(a_t^{m*} + c_e + c_0 - y_t \leq u_t < \infty; x_t)|x_t\right](c_e + c_0). && (142)
\end{aligned}$$

Using the estimates of the monopolist's optimal quotes and the transformation rules for the private value distibution evaluated at the parameter estimates from above delivers the ingredients for the closed form solution of the monopolist's gains from trade. Precisely, applying (111) to (139) delivers

$$\begin{aligned}
& \mathbb{E}\left[I^{m,sell}(-\infty < u_t \leq \hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t; x_t)u_t|x_t\right] \\
&= \hat{\rho}\left(\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) - \hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{-\infty}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right) - \hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{-\infty}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right) \\
&= \hat{\rho}\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) \\
&\quad +(1-\hat{\rho})\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right) && (143)
\end{aligned}$$

Applying (112) to (140) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I^{m,sell}(-\infty < u_t \leq \hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t; x_t)|x_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) - \Phi\left(\frac{-\infty}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right) - \Phi\left(\frac{-\infty}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right) \\
&= \hat{\rho}\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)
\end{aligned} \tag{144}
$$

Applying (111) to (141) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I^{m,buy}(\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t \leq u_t < \infty; x_t)u_t|x_t\right] \\
&= \hat{\rho}\left(\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\infty}{\hat{\sigma}_1^*(y_t, x_t)}\right) - \hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\infty}{\hat{\sigma}_2^*(y_t, x_t)}\right) - \hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right) \\
&= -\hat{\rho}\hat{\sigma}_1^*(y_t, x_t)\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right) \\
&\quad -(1-\hat{\rho})\hat{\sigma}_2^*(y_t, x_t)\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)
\end{aligned} \tag{145}
$$

Applying (112) to (142) delivers

$$
\begin{aligned}
&\mathbb{E}\left[I^{m,buy}(\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t \leq u_t < \infty; x_t)|x_t\right] \\
&= \hat{\rho}\left(\Phi\left(\frac{\infty}{\hat{\sigma}_1^*(y_t, x_t)}\right) - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(\Phi\left(\frac{\infty}{\hat{\sigma}_2^*(y_t, x_t)}\right) - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right) \\
&= \hat{\rho}\left(1 - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_1^*(y_t, x_t)}\right)\right) \\
&\quad +(1-\hat{\rho})\left(1 - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right)\right)
\end{aligned} \tag{146}
$$

Substituting (143) through (146) into (139) through (142) delivers the closed form solution for the estimate of the monopolist's gains from trade:

$$
\begin{aligned}
&\hat{MoG}(x_t) \\
&= \left| -\hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) - (1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right| \\
&\quad - \left(\hat{\rho}\Phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)(\hat{c}_e+\hat{c}_0) \\
&\quad + \left| -\left(\hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right| \\
&\quad - \left(\hat{\rho}\left(1-\Phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\left(1-\Phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right) \\
&\qquad (\hat{c}_e+\hat{c}_0). \\
&= \hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) \\
&\quad - \left(\hat{\rho}\Phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)(\hat{c}_e+\hat{c}_0) \\
&\quad + \hat{\rho}\hat{\sigma}_1^*(y_t,x_t)\phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right) + (1-\hat{\rho})\hat{\sigma}_2^*(y_t,x_t)\phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) \\
&\quad - \left(\hat{\rho}\left(1-\Phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_1^*(y_t,x_t)}\right)\right) + (1-\hat{\rho})\left(1-\Phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right)\right)\right) \\
&\qquad (\hat{c}_e+\hat{c}_0). \qquad (147)
\end{aligned}
$$

Substituting $y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})$ for $\hat{\sigma}_1^*(y_t,x_t)$ and $y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})$ for $\hat{\sigma}_2^*(y_t,x_t)$ as well as $\hat{c}_e^p\cdot y_t$ for $\hat{c}_e$ and $\hat{c}_0^p\cdot y_t$ for $\hat{c}_0$, equation (147) can be written as

$$
\begin{aligned}
&\hat{MoG}(x_t) \\
&= \hat{\rho}y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right) \\
&\quad - \left(\hat{\rho}\Phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{b}_t^{m*}-\hat{c}_e-\hat{c}_0-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)(\hat{c}_e^p+\hat{c}_0^p)y_t \\
&\quad + \hat{\rho}y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{\hat{\sigma}_2^*(y_t,x_t)}\right) \\
&\quad - \left(\hat{\rho}\left(1-\Phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{y_t\hat{\sigma}_1\exp(x_t'\hat{\Gamma})}\right)\right) + (1-\hat{\rho})\left(1-\Phi\left(\frac{\hat{a}_t^{m*}+\hat{c}_e+\hat{c}_0-y_t}{y_t\hat{\sigma}_2\exp(x_t'\hat{\Gamma})}\right)\right)\right) \\
&\qquad (\hat{c}_e^p+\hat{c}_0^p)y_t. \qquad (148)
\end{aligned}
$$

such that the estimate of the monopoly gains from trade as a percentage of the common value equals

$$
\begin{aligned}
&\frac{\hat{MoG}(x_t)}{y_t} \\
&= \hat{\rho}\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{y_t\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{y_t\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})}\right) \\
&\quad - \left(\hat{\rho}\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{y_t\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})\Phi\left(\frac{\hat{b}_t^{m*} - \hat{c}_e - \hat{c}_0 - y_t}{y_t\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})}\right)\right)(\hat{c}_e^p + \hat{c}_0^p) \\
&\quad + \hat{\rho}\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{y_t\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})}\right) + (1-\hat{\rho})\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})\phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{\hat{\sigma}_2^*(y_t, x_t)}\right) \\
&\quad - \left(\hat{\rho}\left(1 - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{y_t\hat{\sigma}_1 \exp(x_t'\hat{\Gamma})}\right)\right) + (1-\hat{\rho})\left(1 - \Phi\left(\frac{\hat{a}_t^{m*} + \hat{c}_e + \hat{c}_0 - y_t}{y_t\hat{\sigma}_2 \exp(x_t'\hat{\Gamma})}\right)\right)\right) \\
&\qquad (\hat{c}_e^p + \hat{c}_0^p). \qquad (149)
\end{aligned}
$$

A.3 Prototypical Programming

Table 1 provides an exemplary programming schedule indicating the data and the tools needed to actually implement the estimation procedure for the Xetra data.

Table 1: **Prototypical Programming**

Program Description	Input	Output	Prototype
SAS program to generate event data set from raw data set delivered by electronic LOB system	Xetra raw data	Xetra event data	`{Xetra event data}` `=generate_LOB_event_data` `(Xetra raw data)`
GAUSS program to estimate competing risks model for latent cancelation and execution times	Xetra event data	Weibull parameter estimates	`{Weibull parameter estimates}` `=MLE_cancelation_execution_times` `(Xetra event data)`
GAUSS program to compute estimates of execution probabilities	Xetra event data, Weibull parameter estimates	Execution probability estimates	`{execution probability estimates}` `=execution_probability_estimates` `(Xetra event data,` `Weibull parameter estimates)`
GAUSS program to estimate regression models for common value changes and to compute estimates of common value changes	Xetra event data	Common value change estimates	`{common value change estimates}` `=OLS_common_value_changes` `(Xetra event data)`
GAUSS program to compute estimates of picking off risks	Xetra event data, common value change estimates	Picking off risk estimates	`{picking off risk estimates}` `=picking_off_risk_estimates` `(Xetra event data,` `common value change estimates)`

Table 1: **Prototypical Programming**

Program Description	Input	Output	Prototype
GAUSS program to evaluate threshold functions at first step estimates	Threshold functions, execution probability estimates, picking off risk estimates	Threshold functions at estimates	`{threshold functions at estimates} =threshold_functions_at_estimates (threshold functions, execution probability estimates, picking off risk estimates)`
GAUSS program to estimate competing risks model for timing of market and limit orders	Xetra event data, threshold functions at estimate	Trader arrival rate estimates, private value distribution estimates, transaction costs estimates	`{trader arrival rate estimates, private value distribution estimates, transaction costs estimates} =MLE_timing_of_MOs_LOs (Xetra event data, threshold functions at estimates)`
GAUSS program to compute estimates of market efficiency measures I and II	Xetra event data, parameter estimates	market efficiency measure I, market efficiency measure II	`{market efficiency measure I, market efficiency measure II} =market_efficiency_measures (Xetra event data, parameter estimates)`

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