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A Dynamic Programming Approach
to Constrained Portfolios[*]

Holger Kraft[1] and Mogens Steffensen[2]

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Abstract

This paper studies constrained portfolio problems that may involve constraints on the probability or the expected size of a shortfall of wealth or consumption. Our first contribution is that we solve the problems by dynamic programming, which is in contrast to the existing literature that applies the martingale method. More precisely, we construct the non-separable value function by formalizing the optimal constrained terminal wealth to be a (conjectured) contingent claim on the optimal non-constrained terminal wealth. This is relevant by itself, but also opens up the opportunity to derive new solutions to constrained problems. As a second contribution, we thus derive new results for non-strict constraints on the shortfall of inter-mediate wealth and/or consumption.

JEL Classifications: G11

Keywords: Finance, Markov Processes, Consumption-investment Problems, Utility Maximization, Bellman Equations

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1 Introduction

Classical dynamic portfolio optimization is concerned with solving non-constrained portfolio problems (see, e.g., Merton (1990)). In practice, a lot of realistic portfolio problems however involve constraints on wealth and consumption. This is because, for instance, financial institutions hold assets to support their obligations to contract holders and to satisfy other stakeholders. Particular examples of these financial institutions are pension funds that we use as a stylized example in this paper.

The objective of this paper is twofold: First, we make a methodological contribution by solving these constrained portfolio problems applying dynamic programming. The standard approach to dynamic portfolio optimization with constraints on wealth is the so-called martingale method. The martingale method was developed by Karatzas et al. (1987) and Cox and Huang (1989) as an alternative to dynamic programming. The method decomposes the dynamic optimization problem into a static optimization problem and a dynamic hedging problem where the latter one is usually involved.[1] On the contrary, dynamic programming gives easy access to the value function and the controls of the problem and thus plays an important role for solving stochastic control problems in finance. To the best of our knowledge, problems with *constraints on wealth* have not been analyzed by dynamic programming yet.[2] Our paper closes this gap and shows how to set up Hamilton-Jacobi-Bellman equations for problems with constraints on both consumption and wealth. We demonstrate how to solve the corresponding highly non-linear partial differential equations. From a stochastic control point of view, this is an important contribution by itself.

Furthermore, and this is our second main contribution, the dynamic programming approach also opens up the opportunity to solve constrained portfolio problems beyond the ones addressed in the literature so far. We are able to study new problems with constraints on intermediate consumption and/or wealth. This is possible because, in contrast to the martingale approach, dynamic programming does not introduce a static optimization problem that is decoupled from the corresponding dynamic hedging problem. For instance, we generalize

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[1]Formally, this is because the martingale representation theorem only guarantees the existence of an optimal portfolio strategy, but does not provide guidance on how to construct a solution.

[2]Notice that wealth is a controlled process. Therefore, the problem is different from studying constraints on controls such as portfolio strategies. We will discuss this point in detail below.

the terminal utility problem considered by Basak and Shapiro (2001) to include intermediate consumption. Here, the fundamental ideas are adapted from Lakner and Nygren (2006), but since we allow for non-strict constraints the results are new. We introduce several ways to relax constraints on intermediate consumption. We formalize a problem where lump sum consumption at discrete time points is restricted by a value-at-risk (VaR) or an expected shortfall constraint. Another situation of special interest for pension asset managers is the case where there is no utility from (but still constraints on) intermediate consumption. All these problems can be addressed with our approach.

Our results give guidance on how to allocate funds among assets that serve a dual purpose: On the one hand, the cash-flows from these assets go to stakeholders of an insurance company. This is described via a goal function that involves *utility of consumption* and/or *wealth*. On the other hand, the assets also protect future obligations (e.g. claims of policy holders) that are modeled via *constraints on* consumption and/or wealth. Problems with no constraints/utility on/from intermediate consumption (Basak and Shapiro (2001)) and with strict constraints on intermediate consumption and/or wealth (Lakner and Nygren (2006)) are included as special cases. There exists an extensive literature on various types of constrained portfolio problems. In general, one can distinguish between two different types of constraints: constraints on terminal wealth ("controlled process") or constraints on portfolio strategies ("controls"). In this paper, we focus on problems with constraints on wealth and also add constraints on intermediate consumption.[3] We however abstract from constraints on portfolio strategies that were extensively studied in recent papers. Furthermore, the literature on consumption-portfolio optimization can also be distinguished w.r.t. the goal function of the problem. In particular, there are papers considering problems with utility maximization, whereas others study problems with classical criteria such as mean-variance maximization. Both problems are relevant, but have to be addressed by applying different methods.[4] Our paper concentrates on utility maximization. Finally, the work in this area can be distinguished w.r.t. whether martingale or dynamic programming methods are applied. As mentioned above, we establish a dynamic programming method to study

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[3]Depending on whether consumption is modeled as lump sum payments or as a continuous stream it can be interpreted as part of the goal function or as control. In this paper, we model consumption as lump sum payments, which is the more realistic case for insurance companies.

[4]For instance, it has been realized that continuous-time mean-variance problems are time-inconsistent, which is in contrast to utility-maximization problem, see Basak and Chabakauri (2010).

consumption-portfolio problems with constraints on intermediate consumption and wealth. Both the method and some of the problems are new (e.g. non-strict constraints on consumption) and contribute to the existing literature. In the remainder of this section, we give a brief overview of this literature.

Constraints on terminal wealth. Grossman and Zhou (1996), Tepla (2001) and Korn (2005) study optimization problems with strict downward constraints on wealth.[5] Basak and Shapiro (2001) consider both relaxed downward constraints that can be violated with a certain probability (VaR constraints) and a constraint where the expected tail loss is restricted (expected shortfall constraint). Basak et al. (2006) and Boyle and Tian (2007) generalize the results for VaR constraints to the case where wealth must exceed a stochastic, but hedgeable, benchmark with a given probability. Korn and Wiese (2008) study the case where, essentially, the benchmark for the portfolio is a non-hedgeable insurance claim, but restrict to certain classes of portfolios with different types of homogeneity assumptions. All these papers use the martingale method[6] and, for instance, do not allow for constraints on intermediate consumption. *Constraints on intermediate consumption and wealth* are usually disregarded in portfolio insurance problems. An exception is Lakner and Nygren (2006) where not only the terminal wealth but also a continuous consumption rate is restricted downwards in a strict sense. As all other above-mentioned papers with constraints on wealth, Lakner and Nygren (2006) use the martingale approach. We distinguish ourselves by using dynamic programming and by allowing for non-strict constraints.

Constraints on portfolio strategies. Firstly, there are papers studying utility maximization problems with portfolio constraints. The classical reference is Cvitanic and Karatzas (1992) who apply duality methods to solve problems with cone constraints.[7] These papers disregard constraints on terminal wealth or intermediate consumption. Furthermore, there is an extensive body of research on the classical mean-variance problem that was originally developed for a static setting, but can be studied in a continuous-time dynamic setting as shown by Zhou and Li (2000). This problem can be combined with constraints on portfolio weights. Typically, such constraints are non-convex and computational methods have to be applied.

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[5]See also Jensen and Sørensen (2001) for an application relevant for the above-mentioned pension fund managers.

[6]The only exception is Korn and Wiese (2008), but they face a different type of optimization problem.

[7]This is a generalization of the martingale method. See, e.g., Cvitanic and Zapatero (2004), Section 4.4, for further details and additional references.

Anagnostopoulos and Mamanis (2008) and Branke et al. (2009) use evolutionary algorithms to search for optimal constrained portfolios in a mean-variance framework. Zhu et al. (2011) apply the particle swarm optimization approach to mean-variance portfolio optimization. Crama and Schyns (2003) solve constrained mean-variance problems by means of simulated annealing. These papers use so-called heuristic optimization methods in order to circumvent the challenges of non-convexity. Special cases of borrowing constraints have been solved by dynamic programming methods.[8] Besides, Emmer, Klüppelberg, and Korn (2001) show that portfolio-insurance-like strategies arise under a quadratic criterion. Osorio et al. (2008) show that a different type of constraints is relevant if mean-variance optimization of post-tax wealth in non-linear tax regimes is analyzed. We distinguish ourselves by working with convex constraints on wealth and consumption rather than portfolio strategies, by working with utility optimization rather than mean-variance optimization, and by working with dynamic programming.

The outline of the paper is as follows: Section 2 presents a general one-period problem and derives a sufficient condition for presenting the solution to an involved (constrained) investment problem as a contingent claim on the solution to a simple (unconstrained) investment problem. Section 3 exemplifies our results from Section 2 and derives the optimal portfolios for a simple linear case, a VaR constraint, and an expected shortfall constraint, respectively. Sections 4 and 5 generalize to intermediate consumption with constraints and to intermediate constraints on wealth. Some proofs can be found in the appendix.[9]

2 The Portfolio Problem and its Value Function

In this section, we relate the solution to an unconstrained portfolio problem to the solution of an involved constrained portfolio problem. We study the decisions of an investor (asset manager) operating in a standard Black-Scholes financial market with two assets, a bond (B) and a stock (S) the dynamics of which are given by

$$dB_t = rB_t dt, \quad B_0 = 1, \qquad dS_t = S_t \left(\alpha dt + \sigma dW_t \right), \quad S_0 = s_0 > 0,$$

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[8]See, e.g., Fu et al. (2010) who allow for non-equal borrowing and lending interest rates.

[9]Longer versions of these proofs are available from the authors upon request.

where r, α and σ are constants. The proportion of assets held in stocks is denoted by π such that the dynamics of the investor's wealth read

$$dX_t = (r + \pi_t (\alpha - r)) X_t dt + \pi_t \sigma X_t dW_t,$$

where $X_0 = x_0 > 0$ denotes his initial wealth. Unless otherwise stated, the investor is assumed to maximize expected utility from terminal wealth with respect to a power utility function $u(x) = x^\gamma/\gamma$, $\gamma < 1$, so that his value function (indirect utility) is

$$V(t, x) = \sup_{\pi \in \mathcal{A}} \mathrm{E}\left[u(X_T) \middle| X_t = x \right],$$

where $\mathcal{A}$ denotes the set of admissible controls. The set $\mathcal{A}$ can be restricted to capture constraints such as a VaR constraint on terminal wealth

$$P(X_T \leq K) \leq \epsilon, \tag{1}$$

where K and ϵ are constants. The value function is characterized by the so-called Hamilton-Jacobi-Bellman (HJB) equation that is given by

$$0 = \inf_{\pi} \left\{ V_t + V_x (r + \pi (\alpha - r)) x + 0.5 V_{xx} \pi^2 \sigma^2 x^2 \right\}, \qquad V(T, x) = u(x). \tag{2}$$

Minimizing over π, the optimal control can be expressed in terms of the value function

$$\pi = -\frac{V_x}{V_{xx} x} \frac{\theta}{\sigma},$$

where $\theta = (\alpha - r)/\sigma$ denotes the market price of risk. Substituting this control into (2) yields the non-linear PDE

$$0 = V_t + V_x r x - 0.5 \theta^2 \frac{V_x}{V_{xx}}, \qquad V(T, x) = u(x). \tag{3}$$

Without additional constraints, Merton (1969, 1971) shows that the solution can be written in form of the separation

$$V(t, x) = u(x)v(t), \tag{4}$$

where v is a deterministic function with $v(T) = 1$. This implies that the optimal stock proportion simplifies into $\pi = \theta/((1 - \gamma)\sigma)$. Consequently, the investor's optimal wealth dynamics without constraints are given by

$$dY_t = Y_t \left(\left(r + \frac{\theta^2}{1-\gamma} \right) dt + \frac{\theta}{1-\gamma} dW_t \right), \qquad Y_0 = y_0 > 0.$$

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The goal of our paper is to study portfolio problems where additional constraints on wealth and/or consumption such as (1) are imposed. In these cases, the above separation breaks down and thus finding the right conjecture for V is involved. For this reason, we suggest a different approach that reduces the dimension of the problem. It turns out that in many relevant applications involving constraints the investor's optimal terminal wealth can be expressed as an option-like contract on his unconstrained optimal wealth Y. Hence, we introduce an option (syn. claim) f on Y and relate its price Π to the solution of the HJB equation. We show that the problem simplifies to finding the *one-dimensional* function f (instead of finding the *two-dimensional* function V). The investor's time-t wealth that corresponds to the claim f is given by the claim price

$$\Pi(t,y) = \mathrm{E}^Q_{t,y}[f(Y_T)]e^{-r(T-t)}, \tag{5}$$

which satisfies a classical Black-Scholes partial differential equation

$$\Pi_t(t,y) = r\Pi(t,y) - ry\Pi_y(t,y) - 0.5\left(\tfrac{\theta}{1-\gamma}\right)^2 y^2\Pi_{yy}(t,y), \qquad \Pi(T,y) = f(y).$$

The initial value y_0 for the process Y is determined as the solution to the equation

$$\Pi(0,y_0) = x_0, \tag{6}$$

i.e. the option price exactly equals the initial wealth of the investor. The following theorem shows how the value function is related to the guess on the claim f and provides a condition under which this guess is correct. To formulate the result, we define the investor's utility of the claim f by

$$U(t,y) = \mathrm{E}_{t,y}[u(f(Y_T))]. \tag{7}$$

The proof of a generalized version of this theorem can be found in Appendix A.

Theorem 1 (Representation of Value Function and Control) *(i) If the condition*

$$-\frac{yU_{yy}}{U_y} = -\frac{y\Pi_{yy}}{\Pi_y} + 1 - \gamma \tag{8}$$

is satisfied, then the value function is characterized by $V(t,\Pi(t,y)) = U(t,y)$. (ii) The optimal stock proportion is characterized by

$$\pi^*(t,y) = \frac{1}{1-\gamma}\frac{y\Pi_y(t,y)}{\Pi(t,y)}\frac{\theta}{\sigma}.$$

Remarks. a) Both the value function V and the optimal stock demand π^* depend on time and wealth. Assuming existence, we define the inverse function of Π with respect to y by Π^{-1}. Then we can express V and π^* as functions of t and x,

$$V\left(t,x\right) = U\left(t, \Pi^{-1}\left(t,x\right)\right), \qquad \pi^*(t,x) = \frac{1}{1-\gamma}\frac{\Pi^{-1}(t,x)\Pi_y(t,\Pi^{-1}(t,x))}{x}\frac{\theta}{\sigma}.$$

b) One can check that condition (8) is satisfied if there exists some function h such that

$$U_y\left(t,y\right) = h\left(t\right)y^{\gamma-1}\Pi_y\left(t,y\right). \tag{9}$$

c) Notice that, in the unconstrained case, we have that $f(y) = y$ so that $\Pi(t,y) = y$. The ratio $y\Pi_y(t,y)/\Pi(t,y)$ is thus equal to one. Therefore, the result by Merton (1969, 1971) follows.

To understand condition (8), recall that for a utility function $u(x)$ the ratio $RRA = -xu_{xx}/u_x$ is said to be the relative risk aversion. Therefore, (8) is satisfied if U has an RRA of $1-\gamma$ and Π has an RRA of 0. This is very natural, since U is a value function induced by a power utility function with RRA of $1-\gamma$. Besides, Π can be interpreted as the value function of a risk-neutral investor who has an RRA of 0, since $\gamma = 1$ for a risk-neutral investor. Without constraints (8) immediately follows because then (4) holds and $f(y) = y$. If however f is not the identity function, but a option-like payoff (as in our applications), then (8) has to be checked separately. Put differently, (8) tells us that the relation between the relative risk aversions must be preserved even if constraints are imposed.

3 Constraints on Terminal Wealth

To make the reader familiar with our approach, we first study constraints on terminal wealth. The most prominent ones are the ones stemming from CPPI-strategies or VaR-constraints. CPPI (constant proportion portfolio insurance) as well as OBPI (option-based portfolio insurance) are actively used by asset managers who need down-side protection.

3.1 CPPI

A utility function that can be used to implement a CPPI-strategy is a power utility function with constant habit level K, i.e.

$$u(x) = \tfrac{1}{\gamma}(x - K)^{\gamma}. \tag{10}$$

This could be the utility functional of an investor who has an obligation of K euros at the investment horizon T. He measures his satisfaction in terms of the utility of the surplus $X - K$. Compared to what follows, the resulting portfolio problem is simpler since the constraint will be satisfied by construction of the utility function (infinite marginal utility at K). This is sharp contrast to the VaR constraint where the constraint is not directly implied by the investor's utility function.

Now, we must find the claim function f and check that condition (8) in Theorem 1 is satisfied. Since the argument of the power utility function is linear in wealth, we conjecture that the optimal terminal wealth can be represented as a linear function of the unconstrained optimal wealth, $f(y) = y + K$, and obtain the following result.

Proposition 2 (Optimal Portfolio for CPPI) *If the investor maximizes expected utility from terminal wealth with respect to the power utility function with habit level (10), then his value function has the representations*

$$V(t, \Pi(t,y)) = \tfrac{1}{\gamma}y^{\gamma}g(t) \qquad or \qquad V(t,x) = \tfrac{1}{\gamma}(x - Ke^{-r(T-t)})^{\gamma}g(t),$$

where $\Pi(t,y) = y + Ke^{-r(T-t)}$. Thus, his optimal stock demand is given by

$$\pi^*(t,y) = \frac{1}{1-\gamma}\frac{y}{y + Ke^{-r(T-t)}}\frac{\theta}{\sigma} \qquad or \qquad \pi^*(t,x) = \frac{1}{1-\gamma}\frac{x - Ke^{-r(T-t)}}{x}\frac{\theta}{\sigma}.$$

Proof. Consider

$$
\begin{aligned}
\Pi(t,y) &= \mathrm{E}_{t,y}^Q\left[e^{-r(T-t)}f(Y_T)\right] = \mathrm{E}_{t,y}^Q\left[e^{-r(T-t)}(Y_T + K)\right] = y + Ke^{-r(T-t)}, \\
U(t,y) &= \mathrm{E}_{t,y}\left[u(f(Y_T))\right] = \mathrm{E}_{t,y}\left[\tfrac{1}{\gamma}((Y_T + K) - K)^{\gamma}\right] = \tfrac{1}{\gamma}y^{\gamma}g(t)
\end{aligned}
$$

with $g(t) = \exp\left(\gamma\left(r - \tfrac{1}{2}\frac{\theta^2}{1-\gamma}\right)(T - t)\right)$. This leads to the partial derivatives $\Pi_y = 1$ and $U_y = y^{\gamma-1}g(t)$, such that condition (9) is satisfied for $h = g$. Finally, y_0 is determined by the relation $x_0 = \Pi(0, y_0) = y_0 + Ke^{-rT}$. This completes the proof. □

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The optimal portfolio does not come as a surprise. The investor should hedge K, which at time t costs $Ke^{-r(T-t)}$. The residual amount $X_t - Ke^{-r(T-t)}$ is invested in Y meaning that the proportion $\frac{\theta}{1-\gamma}$ of $X_t - Ke^{-r(T-t)}$ is invested in stocks and the residual proportion $1 - \frac{\theta}{1-\gamma}$ is invested in bonds. The portfolio is automatically downwards protected by $Ke^{-r(T-t)}$ due to infinite marginal utility at K. This investment strategy is called constant proportion portfolio insurance (CPPI). The surplus $X_t - Ke^{-r(T-t)}$ is said to be the cushion and its proportion invested in stocks (in our case optimally determined to be $\frac{\theta}{1-\gamma}$) is called the multiplier. Due to the linearity of f in this subsection, the solution is particularly simple and has been found by others without explicitly referring to constraints, see e.g. Çanakoğlu and Özekici (2010).

3.2 VaR Constraint

We now consider the problem of maximizing expected utility with a constraint on the probability of terminal wealth being smaller than a given tolerance level. Thus, we consider the problem

$$\sup_{\pi : P(X_T < K) \leq \varepsilon} \mathrm{E}\left[u\left(X_T\right)\right]. \tag{11}$$

For instance, this could be the optimization problem of an asset manager measuring the stakeholders' satisfaction in terms of the utility of the whole asset position at time T. The future obligation K at time T is now taken care of by a constraint on terminal wealth. Whereas in the previous subsection the constraint was build into the utility function, this is not the case in this application. Thus, we deal with the constraint by introducing a Lagrange multiplier λ_ϵ and define the auxiliary utility function

$$\widetilde{u}\left(x\right) = \tfrac{1}{\gamma}x^\gamma - \lambda_\varepsilon \mathbf{1}_{\{x < K\}}. \tag{12}$$

[INSERT FIGURE 1 ABOUT HERE]

Figure 1 illustrates the auxiliary utility function of this problem. The black line is the original utility function without constraints ($\gamma = 0.5$). The red line is the auxiliary utility function including the Lagrange term ($\lambda_\varepsilon = 1, K = 5$). This auxiliary utility function is not concave. However, the straight line (part of the blue line) concavifies this auxiliary utility function and connects linearly the points $\left(5; \tfrac{1}{\gamma}5^\gamma\right)$ and $\left(k_\varepsilon; \tfrac{1}{\gamma}k_\varepsilon^\gamma\right)$. If now the optimal wealth process

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for the concavified auxiliary utility function never ends up in the concavification area, then this is clearly also the optimal solution for the non-concavified utility function. Thus, we need to make sure that the optimal wealth never ends up in the concavification area.

We now have to conjecture the form of f in terms of y. Recall that the VaR risk measure is connected with a confidence level. If this level were 100%, then a VaR constraint can only be satisfied given that the investor protects the portfolio by buying an insurance with full coverage against downside risk. This can be achieved by buying a put option on his terminal wealth. If the level is smaller than 100%, then extreme losses are not considered as to be relevant. Thus, it is natural that the investor sells another put with lower strike than the first put. This suggests the following conjecture

$$f(y) = y + (K - y)\, \mathbf{1}_{\{k_\varepsilon < y < K\}}, \tag{13}$$

where k_ϵ and K can be interpreted as the exercise prices (syn. strikes) of the two options. The functions f and $u(f)$ are illustrated in Figure 2. We see that the optimal wealth, in case our guess is correct, never ends up in the concavification area.

[INSERT FIGURE 2 ABOUT HERE]

To formulate the solution to our portfolio problems, we introduce the following notation: Put(t, y, r, σ, K) denotes the price of a European put option calculated with interest rate r, volatility σ, and strike K, given that the underlying price is y at time t. The expiry date is T. $\Pr_{t,y}^{P}$ denotes the conditional P-probability, given that $Y_t = y$. $\Pr_{t,y}^{Q}$ denotes the conditional Q-probability, given that $Y_t = y$.

Proposition 3 (Optimal Portfolio for VaR-constraints) *If the investor maximizes expected utility from terminal wealth with respect to a power utility function and a VaR constraint (11), then his value function has the representation*

$$
\begin{aligned}
V(t, \Pi(t,y)) &= \tfrac{1}{\gamma} e^{\widetilde{r}(T-t)} \left(y^\gamma + \mathrm{Put}\left(t, y^\gamma, \widetilde{r}, \tfrac{\gamma\theta}{1-\gamma}, K^\gamma\right) - \mathrm{Put}\left(t, y^\gamma, \widetilde{r}, \tfrac{\gamma\theta}{1-\gamma}, k_\varepsilon^\gamma\right) \right) \\
&\quad - \tfrac{1}{\gamma} e^{\widetilde{r}(T-t)} \left(K^\gamma - k_\varepsilon^\gamma + \gamma\lambda_\varepsilon \right) e^{-\widetilde{r}(T-t)} \Pr_{t,y}^{P}\left(Y_T < k_\varepsilon\right),
\end{aligned}
$$

where $\widetilde{r} = \gamma(r + 0.5\theta^2/(1-\gamma))$ and

$$
\begin{aligned}
\Pi(t,y) &= y + \mathrm{Put}\left(t, y, r, \tfrac{\theta}{1-\gamma}, K\right) - \mathrm{Put}\left(t, y, r, \tfrac{\theta}{1-\gamma}, k_\varepsilon\right) \\
&\quad - (K - k_\varepsilon)\, e^{-r(T-t)} \Pr_{t,y}^{Q}\left(Y_T < k_\varepsilon\right).
\end{aligned}
\tag{14}
$$

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His optimal stock demand is given by

$$\pi^*(t,y) = \frac{1}{1-\gamma} \frac{y\Pi_y(t,y)}{\Pi(t,y)} \frac{\theta}{\sigma}.$$

Proof. Consider the value given by (14) derived through

$$\Pi(t,y) = \mathrm{E}_{t,y}^Q \left[e^{-r(T-t)} \left\{ Y_T + (K-Y_T)\mathbf{1}_{\{k_\varepsilon < Y(T) < K\}} \right\} \right] \tag{15}$$

$$= y + \mathrm{E}_{t,y}^Q \left[e^{-r(T-t)} \left\{ (K-Y_T)\mathbf{1}_{\{Y_T < K\}} - (K-Y_T)\mathbf{1}_{\{Y_T < k_\varepsilon\}} \right\} \right]$$

$$= y + \mathrm{Put}\left(t,y,r,\tfrac{\theta}{1-\gamma},K\right) - \mathrm{Put}\left(t,y,r,\tfrac{\theta}{1-\gamma},k_\varepsilon\right) - (K-k_\varepsilon)e^{-r(T-t)}\Pr_{t,y}^Q(Y_T < k_\varepsilon).$$

We now calculate the function $\widetilde{u}(f(y))$

$$\widetilde{u}(f(y)) = \tfrac{1}{\gamma}f(y)^\gamma - \lambda_\varepsilon\mathbf{1}_{\{f(y) < K\}} = \tfrac{1}{\gamma}\left(y + (K-y)\mathbf{1}_{\{k_\varepsilon < y < K\}}\right)^\gamma - \lambda_\varepsilon\mathbf{1}_{\{y < k_\varepsilon\}}.$$

This leads to the following expression for U

$$U(t,y) = \mathrm{E}_{t,y}\left[\widetilde{u}(f(Y_T))\right] = \mathrm{E}_{t,y}\left[\tfrac{1}{\gamma}\left(Y_T + (K-Y_T)\mathbf{1}_{\{k_\varepsilon < Y_T < K\}}\right)^\gamma - \lambda_\varepsilon\mathbf{1}_{\{Y_T < k_\varepsilon\}}\right]$$

$$= \tfrac{e^{\widetilde{r}(T-t)}}{\gamma}\mathrm{E}_{t,y}\left[e^{-\widetilde{r}(T-t)}\left(Y_T^\gamma + (K^\gamma - Y_T^\gamma)\mathbf{1}_{\{k_\varepsilon^\gamma < Y_T^\gamma < K^\gamma\}}\right)\right] - \lambda_\varepsilon\Pr_{t,y}^P(Y_T < k_\varepsilon)$$

$$= \tfrac{e^{\widetilde{r}(T-t)}}{\gamma}\left(y^\gamma + \mathrm{Put}\left(t,y^\gamma,\widetilde{r},\tfrac{\gamma\theta}{1-\gamma},K^\gamma\right) - \mathrm{Put}\left(t,y^\gamma,\widetilde{r},\tfrac{\gamma\theta}{1-\gamma},k_\varepsilon^\gamma\right)\right)$$

$$- \tfrac{e^{\widetilde{r}(T-t)}}{\gamma}\left(K^\gamma - k_\varepsilon^\gamma + \gamma\lambda_\varepsilon\right)e^{-\widetilde{r}(T-t)}\Pr_{t,y}^P(Y_T < k_\varepsilon).$$

Appendix B shows that for $\Pi(t,y)$ and $U(t,y)$ condition (9) holds if $\tfrac{1}{\gamma}K^\gamma - \tfrac{1}{\gamma}k_\varepsilon^\gamma + \lambda_\varepsilon = (K-k_\varepsilon)k_\varepsilon^{\gamma-1}$. This link between λ_ε and k_ε conforms with the concavification argument. It just remains to find λ_ε and hereby k_ε such that $P(X_T < K) = \varepsilon$. Finally, due to the form of claim (13) we must solve two equations in the two unknowns k_ε and y_0 that are given by $P(f(Y_T) < K) = \varepsilon$ and $\Pi(0,y_0) = x_0$. $\qquad\square$

Figure 2 illustrates the optimal asset allocation. The asset manager buys a put option protection for the 'small' losses $k_\varepsilon < Y < K$ and lets the 'big' losses where $Y < k_\varepsilon$ go, such that the shortfall probability tolerance level is used to maximize expected utility. The solution reflects the drawback of the VaR risk measure that it is 'blind' to loss sizes. Therefore, the manager only insures 'small' losses that are cheap to insure and does not protect his portfolio against 'large' losses.

Finally, notice that if the investor has no tolerance for losses, he must buy a full put option protection of Y. This is the special case $\varepsilon = 0$ that corresponds to $\lambda_\varepsilon = \infty$ and has

the solution $k_\varepsilon = 0$. Such a strategy is called an option based portfolio insurance (OBPI) strategy. Although it is not surprising that a put option can protect a portfolio, the important lesson to learn is that it is also the optimal solution to problem (11).

3.3   Expected Shortfall Constraint

We now turn to a different terminal wealth problem and consider the problem of maximizing expected utility with a shortfall constraint on terminal wealth. More precisely, we impose a restriction on the expected shortfall under the risk-neutral measure Q,[10] which essentially means that we work with a tolerance level for the price of this shortfall. This is different from using the physically expected shortfall as risk measure. Rather than limiting the expected shortfall our approach limits the price of the portfolio insurance it would take to protect the optimal portfolio. Therefore, the following problem is studied

$$\sup_{\pi:\mathrm{E}^Q\left[(K-X_T)^+\right]e^{-rT}\leq\varepsilon} \mathrm{E}\left[u\left(X_T\right)\right]. \tag{16}$$

To understand how we should now define the auxiliary utility function $\tilde{u}$, we remark that the VaR constraint can be rewritten as follows:

$$P(X_T \leq K) = \mathrm{E}[\mathbf{1}_{\{X_T\leq K\}}] \leq \epsilon. \tag{17}$$

On the other hand, the expected shortfall constraint can be rewritten as

$$\mathrm{E}^Q\left[(K - X_T)^+ e^{-rT}\right] = \mathrm{E}\left[\mathbf{1}_{\{X_T\leq K\}}\left(K - X_T\right)L_T\right], \tag{18}$$

where L denotes the deflator with dynamics

$$dL_t = L_t\left(-rdt - \theta dW_t\right). \tag{19}$$

Recalling definition (12) and comparing the arguments of the P-expectations in (17) and (18), it is reasonable to define

$$\tilde{u}\left(x,l\right) = \tfrac{1}{\gamma}x^\gamma - \lambda_\varepsilon\mathbf{1}_{\{x<K\}}\left(K - x\right)l.$$

[INSERT FIGURES 3 AND 4 ABOUT HERE]

---

[10]See, e.g., Basak and Shapiro (2001).

Figure 3 depicts the auxiliary utility function of this problem for $l = 1$. The black line is the original utility function without constraints ($\gamma = 0.5$). The red line is the auxiliary utility function including the Lagrange term ($\lambda_\varepsilon = 1, K = 5$). This auxiliary utility is already concave and thus we need no concavification argument. Now, we guess that the claim f on y is given by

$$f(y) = cy\mathbf{1}_{\{y<k_\varepsilon\}} + K\mathbf{1}_{\{k_\varepsilon<y<K\}} + y\mathbf{1}_{\{y>K\}}, \tag{20}$$

where $c = K/k_\varepsilon$. The functions f and $u(f)$ are depicted in Figure 4.

Proposition 4 (Optimal Portfolio for Q-Expected Shortfall constraints) *If the investor maximizes expected utility from terminal wealth with respect to a power utility function and a Q-expected shortfall constraint (16), then his value function has the representation*

$$
\begin{aligned}
V(t,\Pi(t,y),l) &= \tfrac{1}{\gamma}e^{\widetilde{r}(T-t)}\left(y^\gamma + \mathrm{Put}\left(t,y^\gamma,\widetilde{r},\tfrac{\gamma\theta}{1-\gamma},K^\gamma\right) - c^\gamma\mathrm{Put}\left(t,y^\gamma,\widetilde{r},\tfrac{\gamma\theta}{1-\gamma},k_\varepsilon^\gamma\right)\right) \\
&\quad -\lambda_\varepsilon cl\mathrm{Put}\left(t,y,r,\tfrac{\theta}{1-\gamma},k_\varepsilon\right),
\end{aligned}
$$

where

$$\Pi(t,y) = y + \mathrm{Put}\left(t,y,r,\tfrac{\theta}{1-\gamma},K\right) - c\mathrm{Put}\left(t,y,r,\tfrac{\theta}{1-\gamma},k_\varepsilon\right). \tag{21}$$

His optimal stock demand is given by

$$\pi^*(t,y) = \frac{1}{1-\gamma}\frac{y\Pi_y(t,y)}{\Pi(t,y)}\frac{\theta}{\sigma}.$$

Proof. For (21) we obtain

$$\Pi_y(t,y) = 1 + \mathrm{Put}_y(K) - c\mathrm{Put}_y(k_\varepsilon), \qquad \Pi_{yy}(t,y) = \mathrm{Put}_{yy}(K) - c\mathrm{Put}_{yy}(k_\varepsilon), \tag{22}$$

where $\mathrm{Put}(x) = \mathrm{Put}(t,y,r,\tfrac{\theta}{1-\gamma},x)$. One can show that $\widetilde{u}(f(y),l) = \tfrac{1}{\gamma}(cy\mathbf{1}_{\{y<k_\varepsilon\}} + K\mathbf{1}_{\{k_\varepsilon<y<K\}} + y\mathbf{1}_{\{y>K\}})^\gamma - \lambda_\varepsilon c\mathbf{1}_{\{y<k_\varepsilon\}}(k_\varepsilon - y)l$ and thus

$$
\begin{aligned}
U(t,y,l) &= \mathrm{E}_{t,y}\left[\widetilde{u}(f(Y_T),L_T)\right] \\
&= \mathrm{E}_{t,y}\left[\tfrac{1}{\gamma}\left(c^\gamma Y_T^\gamma\mathbf{1}_{\{Y_T^\gamma<k_\varepsilon^\gamma\}} + K^\gamma\mathbf{1}_{\{k_\varepsilon^\gamma<Y(T)^\gamma<K^\gamma\}} + Y_T^\gamma\mathbf{1}_{\{Y_T^\gamma>K^\gamma\}}\right)\right] \\
&\quad -\lambda_\varepsilon c\mathrm{E}_{t,y,l}\left[\mathbf{1}_{\{Y(T)<k_\varepsilon\}}(k_\varepsilon - Y_T)L_T\right] \\
&= \tfrac{e^{\widetilde{r}(T-t)}}{\gamma}\left(y^\gamma + \mathrm{Put}\left(t,y^\gamma,\widetilde{r},\tfrac{\gamma\theta}{1-\gamma},K^\gamma\right) - c^\gamma\mathrm{Put}\left(t,y^\gamma,\widetilde{r},\tfrac{\gamma\theta}{1-\gamma},k_\varepsilon^\gamma\right)\right) - \lambda_\varepsilon cl\mathrm{Put}\left(t,y,r,\tfrac{\theta}{1-\gamma},k_\varepsilon\right).
\end{aligned}
$$

In Appendix C, we show that the functions $\Pi(t,y)$ and $U(t,y,l)$ satisfy the (to L-dependence generalized version of the) conditions in Theorem 1 provided that $\lambda_\varepsilon = y_0^{\gamma-1}e^{\widetilde{r}T}(1 - c^{\gamma-1})$.

Recalling that $c = K/k_\varepsilon$, this is exactly our link between λ_ε and k_ε. Due to the form of the claim (20) we are left with two equations in the two unknowns k_ε and y_0 given by

$$\mathrm{E}^Q \left[e^{-rT} \left(K - f\left(Y_T \right) \right)^+ \right] = \varepsilon \text{ and } \Pi\left(0, y_0 \right) = x_0. \qquad \square$$

Figure 4 illustrates the optimal asset position. As in the previous section, the asset manager buys a put option protection for the 'small' losses $k_\varepsilon < Y < K$, but he now cares also about 'big' losses with $Y < k_\varepsilon$. For these 'big' losses he buys, in addition to Y, an extra linear claim $(c - 1) Y$ such that the claim on Y is continuous at k_ε. Again the special case $\varepsilon = 0$ corresponds to $\lambda_\varepsilon = \infty$ and has the solution $k_\varepsilon = 0$. This corresponds to the OBPI strategy mentioned at the beginning of the section.

4 Intermediate Constraints

4.1 Constraints on Consumption

Until now we have only considered final wealth problems. Now, we allow for intermediate lump sum consumption at discrete time points. To highlight our main ideas, we firstly restrict ourselves to the case of one intermediate consumption date $T_1 \leq T$. This is however without loss of generality. Later on we also explain how these ideas generalizes to multiple periods. For the moment, we consider lump sum consumption C_{T_1} at time T_1: We measure its utility and allow for constraints such as VaR or Q-expected shortfall constraints, e.g. $P\left(C_{T_1} < K_1 \right) \leq \varepsilon_1$. We present the arguments only for the case of a VaR constraint. Other constraints can be handled in a similar way.

In portfolio theory, consumption can be implemented either as a continuous stream (consumption rate) or as lump sum payments. We have chosen the second alternative since from a practical point of view it seems to be the more realistic one. For instance, a fund manager has inflows and outflows on a daily basis, so his grid could be daily. Although the dispersion of payments during a day might be random, the *clearing* of the payments might be well approximated by this approach. Furthermore, a consumption rate can be interpreted as the continuous-time limit of lump sum payments. Therefore, by making the grid finer we can also approximate models with consumption rates well.

We are interested in allocating wealth such as to maximize total expected utility under the

constraints that $P\left(C_{T_1} < K_1\right) \leq \varepsilon_1$ and $P\left(X_T < K\right) \leq \varepsilon$. Here X_T is the residual wealth after financing T_1-consumption C_{T_1}. The wealth dynamics now read

$$dX_t = \left(r + \pi_t\left(\alpha - r\right)\right) X_t dt + \pi_t \sigma X_t dW_t - C_t d\mathbf{1}_{\{t \geq T_1\}}, \quad X_0 = x_0.$$

The investor is interested in maximizing time-T_1 utility of consumption and final wealth by choosing an investment strategy and time-T_1 consumption. For VaR constraints on both lump sum consumption and terminal wealth, the optimization problem becomes

$$V\left(x, \varepsilon_1, \varepsilon_2\right) \equiv \sup_{\pi, C : P\left(C_{T_1} < K_1\right) \leq \varepsilon_1, P\left(X_T < K\right) \leq \varepsilon_2} \mathrm{E}\left[w_1 u_1\left(C_{T_1}, L_{T_1}\right) + w u\left(X_T, L_T\right) \middle| X_0 = x_0\right],$$

$$(23)$$

where w_1 and w are the weights on utility of consumption and terminal wealth, respectively. We now decompose this two-period problem into two one-period problems and a one-dimensional maximization problem. The line of arguments is adapted from Lakner and Nygren (2006), but since our constraints are not strict we need to deal with ε_1 and ε_2 in the right way. We start with an admissible pair (π, C) and define $x_1 = \mathrm{E}\left[L_{T_1} C_{T_1}\right]$ as the time-0 value of consumption C_{T_1} and $x_2 = x_0 - x_1$ as the residual initial amount. The T-problem is that of finding

$$V_2\left(x_2, \varepsilon_2\right) \equiv \sup_{\pi : P\left(X_T < K\right) \leq \varepsilon_2} \mathrm{E}\left[w u\left(X_T, L_T\right) \middle| X_0 = x_2\right]. \qquad (24)$$

It has an optimal solution $\pi_2^{\varepsilon_2}$ and an optimal wealth process $X_2^{\varepsilon_2}$, since this is just the original terminal wealth problem with an adjusted initial wealth. Adding the argument ε_2 emphasizes that the optimal solution is a function of ε_2. The T_1-problem is that of finding

$$V_1\left(x_1, \varepsilon_1\right) \equiv \sup_{\pi, C : P\left(C_{T_1} < K_1\right) \leq \varepsilon_1} \mathrm{E}\left[w_1 u\left(C_{T_1}, L_{T_1}\right) \middle| X_0 = x_1\right]. \qquad (25)$$

This problem has an optimal solution $(\pi_1^{\varepsilon_1}, C_1^{\varepsilon_1})$ and an optimal wealth process $X_1^{\varepsilon_1} = X - X_2^{\varepsilon_2}$. Notice that this can be interpreted as a terminal wealth problem terminating at time T_1. Now define $\widetilde{C}$, $\widetilde{X}$, and $\widetilde{\pi}$ by

$$\widetilde{C}^{\varepsilon_1, \varepsilon_2} = C_1^{\varepsilon_1}, \quad \widetilde{X}^{\varepsilon_1, \varepsilon_2} = X_1^{\varepsilon_1} + X_2^{\varepsilon_2}, \quad \widetilde{\pi}^{\varepsilon_1, \varepsilon_2} \widetilde{X}^{\varepsilon_1, \varepsilon_2} = \pi_1^{\varepsilon_1} X_1^{\varepsilon_1} + \pi_2^{\varepsilon_2} X_2^{\varepsilon_2}.$$

Then $\widetilde{X}$ is the wealth process corresponding to the pair $(\widetilde{\pi}, \widetilde{C})$. Since we have formed the solutions for each sub-problem, we can compare the expected utility of these strategies to our original admissible strategy (π, C) with wealth process X. We then know that

$$V_1\left(x_1, \varepsilon_1\right) \geq \mathrm{E}\left[w_1 u\left(C_{T_1}, L_{T_1}\right) \middle| X_0 = x_1\right], \quad V_2\left(x_2, \varepsilon_2\right) \geq \mathrm{E}\left[w u\left(X_T, L_T\right) \middle| X_0 = x_2\right],$$

such that also $V_1\left(x_1, \varepsilon_1\right) + V_2\left(x_2, \varepsilon_2\right) \geq \mathrm{E}\left[u\left(C, L_{T_1}\right) + u\left(X_T, L_T\right) \middle| X\left(0\right) = x_0\right]$. Here we first take maximum on the left hand side over all combinations $\left(x_1, x_2\right)$ adding up to x_0. Second we take supremum over all admissible pairs $\left(\pi, C\right)$ on the right hand side, which yields

$$\max_{x_1, x_2 : x_1 + x_2 = x_0}\left[V_1\left(x_1, \varepsilon_1\right) + V_2\left(x_2, \varepsilon_2\right)\right] \geq V\left(x, \varepsilon_1, \varepsilon_2\right). \tag{26}$$

Notice that the pair $\left(\widetilde{\pi}, \widetilde{C}\right)$ corresponding to x_1 and x_2 achieves the maximum on the left hand side of (26). Besides, following the optimal constrained strategies in the two sub-problems is an admissible strategy itself. Therefore, (26) holds with equality. This implies that the pair $\left(\widetilde{\pi}, \widetilde{C}\right)$ based on x_1 and x_2 solving the maximization problem in (26) is indeed optimal for (23).

The optimal solution $\left(x_1, x_2\right)$ to the maximization problem on the left hand side of (26) is described by the following equation in x_1,

$$\left.\frac{\partial}{\partial x}V_1\left(x_1, \varepsilon_1\right)\right|_{x=x_1} = \left.\frac{\partial}{\partial x}V_2\left(x_2, \varepsilon_2\right)\right|_{x=x_0-x_1}. \tag{27}$$

This condition connects the time-T_1 and the time-T problem via the budget constraint $x_0 = x_1 + x_2$. Now, the full problem reduces to solving three non-linear equations with three unknowns: Two of theses equations are the Lagrange equations setting the Lagrange multipliers such that the two constraints are fulfilled,

$$P\left(C_{T_1} < K_1\right) = \varepsilon_1, \qquad P\left(X_T < K\right) = \varepsilon_2. \tag{28}$$

The third equation is (27). The three unknowns are $\left(k_1^{\varepsilon_1}, k_2^{\varepsilon_2}, x_1\right)$. The intuition of the result is that the total asset allocation decomposes into a capital allocation problem and two terminal wealth asset allocation problems with different time horizons.

The multi-period case. The generalization to a finer grid for lump sum consumption is straight-forward. If we have n time points of intermediate consumption $T_1, \ldots, T_n$ from which we measure power utility with constraints on consumption at each time point, we get $2n + 1$ non-linear equations with $2n + 1$ unknowns: For each intermediate constraint we have one equation that determines the Lagrange multiplier $\lambda_i\left(\varepsilon_i\right)$ such that the constraint $\mathrm{E}\left[f_i\left(C_{T_i}, L_{T_i}\right)\right] \leq \varepsilon_i$ is satisfied. For the terminal constraint we have one equation that determines the Lagrange multiplier $\lambda_{n+1}\left(\varepsilon_{n+1}\right)$ such that the constraint $\mathrm{E}\left[f_{n+1}\left(X_T, L_T\right)\right] \leq \varepsilon_{n+1}$ is fulfilled. This gives the first $n + 1$ equations and $n + 1$ unknowns. Furthermore,

the initial wealth x_0 constrains the wealth allocated to each consumption by the budget constraint $x_1 + \ldots + x_n + x_{n+1} = x_0$ yielding to n equations for n unknowns:

$$\frac{\partial}{\partial x} V_1\left(x, \varepsilon_1\right)\bigg|_{x=x_1} = \ldots = \frac{\partial}{\partial x} V_n\left(x, \varepsilon_n\right)\bigg|_{x=x_n} = \frac{\partial}{\partial x} V_{n+1}\left(x, \varepsilon_{n+1}\right)\bigg|_{x=x_0 - \sum\limits_{i=1}^{n} x_i}.$$

This has clear applications in pension fund management, where the obligations typically are formalized in terms of a cash flow that essentially consists of periodical (e.g. annual) obligations up to 50 or more years into the future.

The multi-constraint one-period case. For the above calculations, it is not necessary that T_1 is strictly smaller than T. In fact, consumption can take place at time T and then the utility of the residual wealth $X_T = X_{T-} - C_T$ is measured. This idea leads to an optimization problem where utility is measured separately for C_T and $X_{T-} - C_T$. Therefore, one can also impose separate constraints on both parts. Similarly, if we have n lump sum consumptions at time T from which utility is measured and on which we have constraints, then all equations above hold true for $T_1 = \ldots = T_n = T$. This pattern of thinking can be applied to pension fund management since the total cash flow at a particular time point is the sum over all payouts to all contract holders. The fund could measure the individual utility of payouts and add them up to measure the total utility of the cash flow at that time point.

The strict constraint case. Now, we consider the case where $\varepsilon_1 = \varepsilon_2 = 0$. This is simple since the two equations (28) have obvious solutions: $k_1^{\varepsilon_1} = k_2^{\varepsilon_2} = 0$. This leaves us with one equation with one unknown. In the general above-discussed multi-period case, there are $n + 1$ constraints and $2n + 1$ equations with $2n + 1$ unknowns. In the strict constraint case, $n + 1$ of these equations have similarly simple solutions, such that we are left with only n equations with n unknowns corresponding to the wealth allocated to consumption at each of the intermediate time points. This is of course much simpler than the non-strict case.

The case of no utility from consumption. Another interesting special case arises if there is no utility from intermediate consumption $(w_1 = 0)$, but intermediate consumption is still constrained. This changes the time-T_1 problem. But the separation of the total problem into two terminal value problems still holds. Since V_2 is increasing in x_2, it is clear that we now have to find the cheapest way to satisfy the probability constraint, i.e. find the payoff C, the investment strategy, and its price such that

$$\arg\inf_{\pi, C : P\left(C_{T_1} < K_1\right) \leq \varepsilon_1} \mathrm{E}\left[C_{T_1} L_{T_1}\right], \qquad x_1 = \inf_{\pi, C : P\left(C_{T_1} < K_1\right) \leq \varepsilon_1} \mathrm{E}\left[C_{T_1} L_{T_1}\right].$$

The residual amount $x_2 = x_0 - x_1$ is then spent on the time-T utility maximization problem. The problem of minimizing the hedging cost for a given shortfall probability was addressed by Föllmer and Leukert (1999). We now give an intuitive argument for their solution in terms of our setup. For the time-T_1 problem the solution is given as the solution to the terminal utility problem for a log-utility investor, since the log utility investor essentially invests so as to hedge a multiple of L. The solution of the log-utility case is found above by setting $\gamma = 0$. So to solve the time T_1 problem we set $\gamma = 0$ and consider the claim (13) on Y with $K = K_1$. So far the final step has been to find the initial point y_0 which is essentially the slope of the linear part of $f(y)$ such that the price of the claim equals a given value. Here, instead, we need to find y_0 such that the value of the claim x_1 is minimized, since this maximizes the residual amount $x_2 = x_0 - x_1$, which again maximizes utility. This slope is zero leading to the claim $f(y) = K\mathbf{1}_{\{k_\varepsilon < y\}}$. The constant k_ε is determined by the constraint $P(C_{T_1} < K_1) = P(Y > k_\varepsilon) = \varepsilon$.

Complicated and interesting problems not addressed here. The solution to the consumption problem involves an allocation of capital at time 0 to the two consumption plans at time T_1 and T. This feature of the solution is not destroyed by the constraints. This connects to the fact that utility of consumption at T_1 and T is linked together by the budget constraint only. In two popular generalizations, however, there is a second link via preferences. For habit formation the utility of consumption at time T depends on the consumption level at time T_1. For recursive utility the utility consumption at time T_1 depends on the consumption level at time T through the value function at time T_1. In these cases, the simple capital allocation at time 0 satisfying the budget constraint is insufficient and one needs to move capital between the consumption plans during the investment period. This is however a considerably more difficult task and it is not addressed here.

4.2 Constraints on Intermediate Wealth

Strictly speaking, Theorem 1 can only be applied to constraints on terminal wealth. However, in the following, we wish to study intermediate constraints such as

$$P(X_{T_1} \leq K_1) \leq \epsilon_1, \tag{29}$$

where $T_1 < T$ and K_1 as well as ϵ_1 are constants. In the light of Theorem 1, this means that we have to deal with constraints on $X_t = \Pi(t, Y_t)$ instead on X_T. For simplicity, we

assume that constraints are imposed at two time points T_1 and T only. For $t \in [T_1, T]$, there is only one constraint and we are in the situation of Theorem 1. Therefore, we can solve for the optimal wealth $X_t^* = \Pi(t, Y_t)$ expressed in terms of the claim value Π. For $t \in [0, T_1]$, Theorem 1 does not directly apply. However, by the Bellman principle, the value function for $t \le T_1$ reads

$$V(t, x) = \sup_\pi \mathrm{E}_{t,x}[V(T_1, X_{T_1})].$$

Therefore, we define for $t \le T_1$

$$\hat{\Pi}(t, y) = \mathrm{E}_{t,y}^Q[\hat{f}(\Pi(T_1, Y_{T_1}))]e^{-r(T_1-t)}, \quad \hat{U}(t, y) = \mathrm{E}_{t,y}[V(T_1, \hat{f}(\Pi(T_1, Y_{T_1})))]$$

for a claim function $\hat{f}$ related to the constraint at time T_1.

Proposition 5 *If $\hat{U}$ satisfies*

$$\frac{1 - \gamma}{y}\hat{U}_y + \hat{U}_{yy} = \frac{\hat{\Pi}_{yy}\hat{U}_y}{\hat{\Pi}_y}, \tag{30}$$

then $\hat{V}(t, \hat{\Pi}(t, y)) = \hat{U}(t, y)$.

Remark. The only difference to Section 2 is that the ordinary option Π is replaced by the compound option $\hat{\Pi}$. Compound options are well-known in finance where they are for instance used to model equity as a call option on firm value, see Geske (1979). From this point of view, a call on equity can be interpreted as a call on a call. In principle, the situation is the same here, but the computations are much more complicated since we are interested in a more general 'derivative on a derivative' where the payoff structures of the derivatives are given in (13). Geske (1979) considers two different maturities, one for the horizon of the firm valuation and one for the maturity of the option on firm value. This is the same here where the underlying derivative has maturity T and the derivative on that derivative has maturity T_1.

Proof. By the Feynman-Kac theorem, the function $\hat{U}$ satisfies the same PDE as U

$$0 = \hat{U}_t + y\mu_y\hat{U}_y + 0.5y^2\sigma_y^2\hat{U}_{yy}$$

with the *generalized boundary condition* $\hat{U}(T_1, y) = V(T_1, \hat{f}(\Pi(T_1, y)))$. Notice that the boundary condition for U can be interpreted as a special case of this boundary condition by formally setting $V(T_1, \hat{f}(\Pi(T_1, y))) = u(f(y))$. The boundary conditions coincide since

$$\hat{U}(T_1, y) = V(T_1, \hat{f}(\Pi(T_1, y))) = V(T_1, \hat{\Pi}(T_1, y)) = \hat{V}(T_1, \hat{\Pi}(T_1, y)).$$

19

We set $\hat{V}^*(t, \hat{\Pi}(t, y)) = \hat{U}(t, y)$. The relevant non-linear PDE is $0 = \hat{V}_t + rx\hat{V}_x - 0.5\theta^2\hat{V}_x^2/\hat{V}_{xx}$ with the *generalized boundary condition* $\hat{V}(T_1, x) = V(T_1, x)$. The compound option satisfies

$$\hat{\Pi}_t - r\hat{\Pi} + ry\hat{\Pi}_y + 0.5y^2\sigma_y^2\hat{\Pi}_{yy} = 0$$

with the *generalized boundary condition* $\hat{\Pi}(T_1, y) = \hat{f}(\Pi(T_1, y))$. If we now use the fact that for $0 \leq t \leq T_1$ the optimal wealth process equals the compound option, i.e. $x = \hat{\Pi}(t, y)$, then the proof works exactly as the proof of Theorem 1. □

5 Conclusion

This paper provides a new approach to solve constrained portfolio problems. We use control theory to construct the value functions to these problems and show how to solve the corresponding highly non-linear partial differential equations. Although important by itself, our approach opens up the opportunity to derive new closed-form solutions. We demonstrate that for non-strict constraints on the shortfall of intermediate wealth and/or consumption. Interesting generalizations might be problems with recursive utility and habit formation. This is left for future research.

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A Proof of Theorem 1

Here we prove Theorem 1 for the generalized case where the deflator L is added as state process. In that case the condition (8) reads

$$-\frac{yU_{yy}}{U_y - U_{ly}l} = -\frac{y\Pi_{yy}}{\Pi_y} + 1 - \gamma. \tag{31}$$

We have to check two conditions, namely a terminal condition and a PDE condition. First, for the terminal condition we calculate from (7), (5), and (3),

$$U\left(T, y, l\right) = u\left(f\left(y\right), l\right) = u\left(\Pi\left(T, y\right), l\right) = V\left(T, \Pi\left(T, y\right), l\right).$$

Second, we define V^* as $V^*\left(t, \Pi\left(t, y\right), l\right) = U\left(t, y, l\right)$. One can then express the derivatives of V in terms of the derivatives of U. Notice that U is characterized by the PDE

$$
\begin{aligned}
U_t\left(t, y, l\right) = &-\left(r + \tfrac{\theta^2}{1-\gamma}\right)yU_y\left(t, y, l\right) - 0.5\left(\tfrac{\theta}{1-\gamma}\right)^2 y^2 U_{yy}\left(t, y, l\right) + rlU_l\left(t, y, l\right) \quad (32)\\
&-0.5\theta^2 l^2 U_{ll}\left(t, y, l\right) + \tfrac{y}{1-\gamma}\theta^2 lU_{ly}\left(t, y, l\right), \qquad U\left(T, y, l\right) = u\left(f\left(y\right), l\right).
\end{aligned}
$$

Finally, one can check that $V^*\left(t, \Pi\left(t, y\right)\right)$ is a candidate for our value function by calculating the right hand side of (3) with V replaced by V^*. We skip the corresponding calculations. $\square$

B Verification for the VaR Constraint

Defining $d_1\left(t, y, k, r, \sigma\right) = \left(\log\left(y/k\right) + \left(r + 0.5\sigma^2\right)\left(T - t\right)\right)/\left(\sigma\sqrt{T - t}\right)$, one can show that $d_1\left(t, y^\gamma, K^\gamma, \widetilde{r}, \tfrac{\gamma\theta}{1-\gamma}\right) = d_1\left(t, y, K, r, \tfrac{\theta}{1-\gamma}\right)$. This implies

$$\frac{\partial}{\partial y}\text{Put}\left(t, y^\gamma, \widetilde{r}, \tfrac{\gamma\theta}{1-\gamma}, K^\gamma\right) = \gamma y^{\gamma-1}\frac{\partial}{\partial y}\text{Put}\left(t, y, r, \tfrac{\theta}{1-\gamma}, K\right). \quad (33)$$

Furthermore, one can show that

$$e^{-r(T-t)}\frac{\partial}{\partial y}\Pr_{t,y}^Q\left(Y_T < k_\varepsilon\right) = \frac{y}{k_\varepsilon}N'\left(d_1\left(t, y, k_\varepsilon, r, \tfrac{\theta}{1-\gamma}\right)\right)d_1'\left(t, y, k_\varepsilon, r, \tfrac{\theta}{1-\gamma}\right),$$

and

$$e^{-\widetilde{r}(T-t)}\frac{\partial}{\partial y}\Pr_{t,y}^P\left(Y_T < k_\varepsilon\right) = \frac{1}{k_\varepsilon^{\gamma-1}}y^{\gamma-1}e^{-r(T-t)}\frac{\partial}{\partial y}\Pr_{t,y}^Q\left(Y_T < k_\varepsilon\right). \quad (34)$$

Now, substituting (33) and (34) into the derivatives of (15) and (16), one gets the relation

$$U_y\left(t, y\right) = e^{\widetilde{r}(T-t)}y^{\gamma-1}\Pi_y\left(t, y\right),$$

under the condition $\frac{1}{\gamma}K^\gamma - \frac{1}{\gamma}k_\varepsilon^\gamma + \lambda_\varepsilon = \left(K - k_\varepsilon\right)k_\varepsilon^{\gamma-1}.$ $\square$

C Verification for the Expected Shortfall Constraint

One can calculate the derivatives of U to show that

$$U_y - U_{ly}l = e^{\widetilde{r}(T-t)}y^{\gamma-1}\left(1 + \text{Put}_y\left(K\right) - c^\gamma\text{Put}_y\left(k_\varepsilon\right)\right). \quad (35)$$

We now check condition (31) which is the generalized condition in Theorem 1 if L is a state process. From (35), (22), the derivatives of U and a series of calculations, we get

$$\left(\frac{1-\gamma}{y} \left(U_y - U_{ly} l \right) + U_{yy} \right) \Pi_y - \Pi_{yy} U_y \tag{36}$$
$$= \left(\mathrm{Put}_y \left(k_\varepsilon \right) \mathrm{Put}_{yy} \left(K \right) - \mathrm{Put}_{yy} \left(k_\varepsilon \right) - \mathrm{Put}_{yy} \left(k_\varepsilon \right) \mathrm{Put}_y \left(K \right) \right) \left(\lambda_\varepsilon c l - e^{\widetilde{r}(T-t)} y^{\gamma-1} \left(c - c^\gamma \right) \right).$$

Notice that this does not have to be zero at any point in the state space. Since, almost surely, $L_t = e^{-\widetilde{r}t} \left(Y_t / y_0 \right)^{\gamma-1}$, we only need to check the plane in (t, y, l) given by $l = e^{-\widetilde{r}t} \left(\frac{y}{y_0} \right)^{\gamma-1}$. But then the second factor in (36) can be written as

$$\lambda_\varepsilon c e^{-\widetilde{r}t} \left(\frac{y}{y_0} \right)^{\gamma-1} - e^{\widetilde{r}(T-t)} y^{\gamma-1} \left(c - c^\gamma \right) = y^{\gamma-1} c e^{-\widetilde{r}t} \left(\lambda_\varepsilon \left(\frac{1}{y_0} \right)^{\gamma-1} - e^{\widetilde{r}T} \left(1 - c^{\gamma-1} \right) \right),$$

which is zero if $\lambda_\varepsilon = y_0^{\gamma-1} e^{\widetilde{r}T} \left(1 - c^{\gamma-1} \right)$. $\qquad\square$
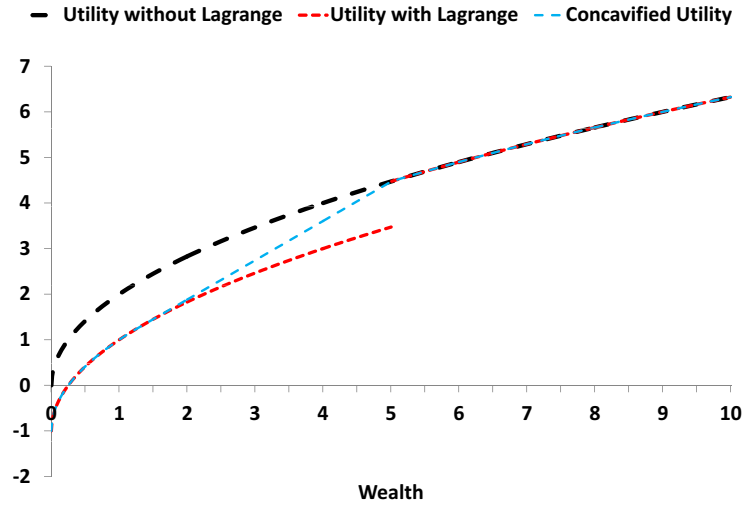
Figure 1: Utility function, utility function with Lagrange term, and concavified utility function for the VaR-constraint case.
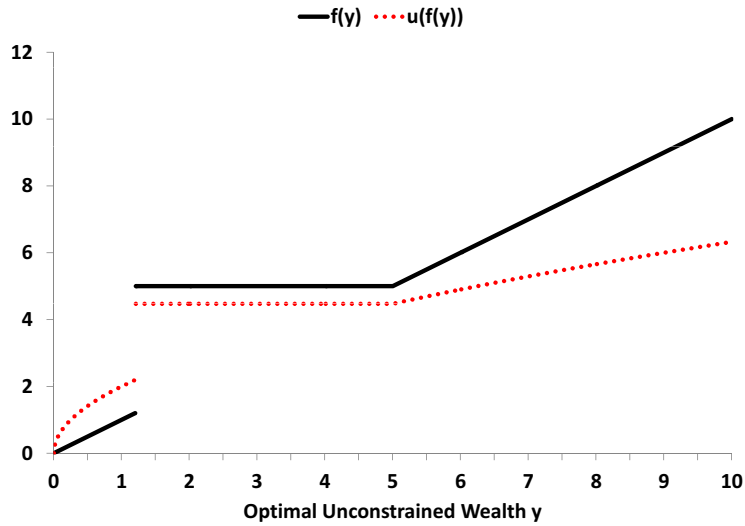

Figure 2: Optimal wealth and utility of optimal wealth as a function of Y for the VaR-constraint case.
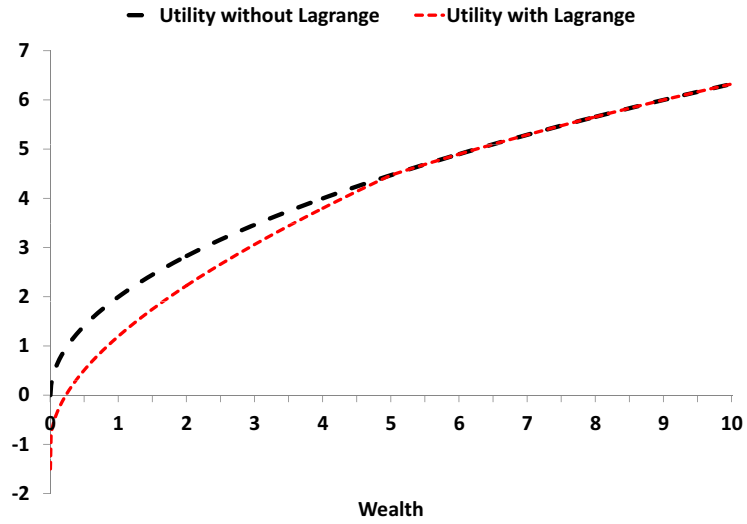
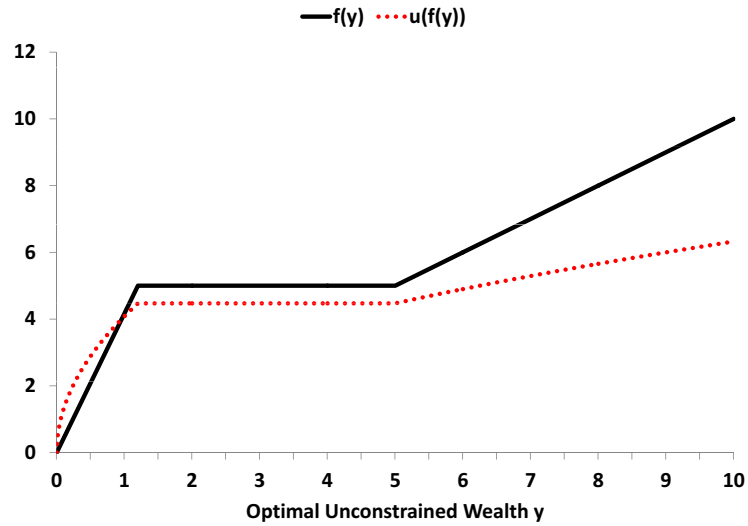Figure 3: Utility function and utility function with Lagrange term for the expected shortfall constraint case.

Figure 4: Optimal wealth and utility of optimal wealth as a function of Y for the expected shortfall constraint case.

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